COUNTING AND SAMPLING ANTI-FERROMAGNETIC POTTS MODELS ON RANDOM REGULAR BIPARTITE GRAPHS IN THE NON-UNIQUENESS REGIME

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Abstract. The anti-ferromagnetic multi-state Potts model, a generalization of the Ising model, is one of the most fundamental models in statistical physics. It was conjectured by Kotecký (Phys. Rev. B, 1985) that the model undergoes a phase transition from a disordered phase at infinite temperature to an ordered phase at sufficiently low temperature on lattices. Such phase transitions are believed to play an important role in computational complexity theory and remain closely connected to the problem of approximating the partition function of the system. For proper three-coloring models (corresponding to the zero-temperature), torpid mixing of a family of local-update Markov chains on lattices was established by Galvin, Kahn, Randall and Sorkin (SIDMA, 2015), coinciding with the presence of phase coexistence following shown by Feldheim and Spinka (J. Eur. Math. Soc., 2019).

In this work, we study approximating the partition function of the anti-ferromagnetic multi-state Potts model at low temperature on random regular bipartite graphs, which are with high probability good bipartite expanders. On the negative side, we generalize the result by Geisler, Kang, Sarantis and Wdowinski (arXiv, 2026) for anti-ferromagnetic Ising models to show that when the temperature is sufficiently low relative to the degree of the underlying graph, the celebrated single-site Glauber dynamics has exponentially slow mixing time. On the positive side, we design a deterministic algorithm that yields an approximation to the partition function of the model via the framework of abstract polymer models as Jenssen, Keevash and Perkins (SICOMP, 2020), Liao, Lin, Lu and Mao (Theor. Comput. Sci., 2022), Galanis, Goldberg and Stewart (TOCT, 2021) and Geisler, Kang, Sarantis and Wdowinski (arXiv, 2026).

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1. INTRODUCTION

The q-state Potts model (q-Potts model, for short), named after R. B. Potts introduced in [Pot52], is a generalization of the Ising model which is one of the fundamental models in statistical physics and has been well studied in probability theory and theoretical computer science. Formally, given a ground state U, the model is specified by a color set $[q] := \{1, \dots, q\}$ for a positive integer $q \geq 3$, an inverse temperature β and a Hamiltonian $H : [q]^U \to \mathbb{R}$. A *configuration* σ is an assignment $U \to [q]$ and the weight of σ is defined by:

$$w(\sigma) := \mathrm{e}^{-\beta H(\sigma)}. \tag{1}$$

Let the state space or configuration space $\Omega := [q]^U$ be the collection of all configurations. When $\beta < 0$, we say that it is a *ferromagnetic q-Potts model* and an *anti-ferromagnetic* case for $\beta > 0$.

We study the *anti-ferromagnetic q-Potts model* on graphic models. Namely, for a graph $G = (V, E)$, we set the ground state $U = V$ (and thus $\Omega = [q]^V$), and define the Hamiltonian $H(\cdot)$ by the number of monochromatic edges in G under a configuration, *i.e.*,

$$\forall \sigma \in \Omega, \quad H(\sigma) = m_G(\sigma) := |\{(u,v) \in E : \ \sigma|_u = \sigma|_v\}|. \tag{2}$$

The Gibbs distribution induced by the anti-ferromagnetic q-Potts model on G at β is the probability distribution $\mu = \mu_{G,q;\beta}$ on Ω defined as

$$\forall \sigma \in \Omega, \quad \mu(\sigma) = \frac{w(\sigma)}{Z_{G,q}(\beta)} = \frac{\mathrm{e}^{-\beta m_G(\sigma)}}{Z_{G,q}(\beta)}$$

where $Z_{G,q}(\beta) = \sum_{\sigma \in \Omega} w(\sigma)$ is the *partition function* of the model. For example, when the temperature is zero ($\beta = \infty$), it holds that $w(\sigma) = 1$ for all configuration σ representing a *proper vertex-coloring* of G and $w(\sigma) = 0$ otherwise. Thus we can view the proper vertex-coloring model as a special case of anti-ferromagnetic q-state Potts models.

A central problem to study the anti-ferromagnetic Potts model is the computation of the partition function $Z_{G,q}(\beta)$. Unfortunately, the exact estimation to $Z_{G,q}(\beta)$ is #**P**-complete, even when the underlying graph is a regular triangle-free graph ([Gre00]). Therefore, a natural problem is arising for computing Potts models:

> *Is there an efficient algorithm giving an approximation to the partition function of the Potts model, either a deterministic one or a randomized one?*

We use the following notations to describe efficient approximate algorithms. For a function Z from a family of instances to reals, a *fully polynomial-time approximation scheme* (**FPTAS**) is an algorithm such that for each instance Φ and a tolerance error $\varepsilon \in (0,1)$, it outputs a number $\widehat{Z}$ in time poly $\left(|\Phi|, \varepsilon^{-1}\right)$[1] satisfying that $\widehat{Z}$ is an ε-approximation to $Z(\Phi)$, *i.e.*, $(1-\varepsilon)Z(\Phi) \leq \widehat{Z} \leq (1+\varepsilon)Z(\Phi)$. Similarly, a *fully polynomial-time randomized approximation scheme* (**FPRAS**) is a randomized algorithm outputting a $\widehat{Z} \in (1 \pm \varepsilon)Z(\Phi)$ in time poly $\left(|\Phi|, \varepsilon^{-1}\right)$ with probability at least $3/4$[2].

An important topic closely related to the design of approximate algorithms is the *uniqueness* of the Gibbs distribution. Consider the parameter $\beta_c(q, \Delta)$ defined as:

$$\beta_c(q, \Delta) := \ln \max \left\{1, \frac{\Delta}{\Delta - q}\right\}.$$

It is a folklore conjecture that the Gibbs distribution on $\mathbb{Z}^\Delta$ is unique if and only if $\beta \leq \beta_c(q, \Delta)$. Such characterization of the uniqueness threshold on the infinite Δ-regular tree (or the *Bethe lattice*) is proved by Galanis, Goldberg and Yang [GGY18] for $q = 3$ and subsequently improved by Bencs

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[1] The cardinality of an instance Φ is the size of its input.

[2] The probability $3/4$ is standard. One can substitute it with any real number $> 1/2$.

et al. [BBBR23] for general $q \geq 3$ but large enough Δ. The tree-uniqueness threshold plays a crucial role in computational theory since it has been conjectured that there exists an efficient approximation (either an **FPTAS** or **FPRAS**) to the partition function of anti-ferromagnetic Potts model if and only if β is in the tree-uniqueness threshold, as in the anti-ferromagnetic Ising model ([LLY13, GŠV16]).

To design an approximate algorithm in the tree-uniqueness threshold, Lu and Yin [LY13] propose an **FPTAS** via the so-called *correlation-decay* method when $3\Delta(\mathrm{e}^{|\beta|} - 1) < 1$ for general Potts models. Afterwards in the study of the zero-free region of the proper vertex-coloring model, Liu, Sinclair and Srivastava [LSS25] establish that the partition function $Z_{G,q}(\beta)$ is not zero when $q \geq 2\Delta$, leading to an **FPTAS** by [PR17]. The result of the zero-freeness region is recently improved by [BBR26] for $q \geq (2-\eta)\Delta$ for $\eta \geq 0.002$. For an efficient sampler, Blanca et al. [BGG$^+$20] provide an algorithm to sample from $\mu_{G,q;\beta}$ on random Δ-regular graphs within total variation distance $n^{-\delta}$ for some constant $\delta = \delta(\Delta, q, \beta)$. Note that the sampler in [BGG$^+$20] itself cannot directly induce any **FPRAS** since it just outputs a configuration within a fixed error.

In the non-uniqueness threshold, Galanis, Štefankovič and Vigoda [GŠV15] show that for even positive integers q and Δ, and $\Delta(1 - \mathrm{e}^{-\beta}) > q$, there exists no **FPRAS** to approximate the partition function unless **NP** = **RP** even on triangle-free Δ-regular graphs. However, despite the hardness result, there might be different cases when the anti-ferromagnetic Potts model is defined on a bipartite graph. When the model is defined on lattice $\mathbb{Z}^\Delta$, it has been conjectured by Kotecký [Kot85] that there exists a phenomenon called *long-range order* for anti-ferromagnetic Potts models. Briefly speaking, when the temperature goes from ∞ to 0, the model undergoes a transition from disordered phase to 'ordered phases'. It is following answered by Feldheim and Spinka [FS19] that such coexistence of different phases occurs for $q = 3$ and sufficiently large Δ at $\beta > \ln \Delta$, which is far away from the tree-uniqueness threshold.

The existence of long-range order might give some inspiration to the design of algorithms. On one hand, Galvin et al. [GKRS15] show that any 'local-updated' Markov chain exhibits an exponential mixing time for proper 3-colorings on $\mathbb{Z}^\Delta$ where the phase transition occurs. On the other hand, Jenssen, Keevash and Perkins [JKP20] shows that for $q = O(\sqrt{\Delta}/(\ln^2 \Delta))$ or equivalently $\Delta = \Omega((q \ln q)^2)$, there exists an **FPTAS** to approximately count the number of proper colorings on random Δ-regular bipartite graphs with high probability, by showing that each configuration follows some kind of 'ordered phase' in proper colorings. Similar techniques are applied to general spin systems (including both ferromagnetic and anti-ferromagnetic Potts models) by Galanis, Goldberg and Stewart [GGS21] to design a Markov chain for the models on random Δ-regular bipartite graphs, with a harder constraint $\Delta/\ln^4 \Delta = \Omega((q \ln q)^4)$ on Δ and q. It still needs a deep understanding on whether the existence of such multiple phases indicates an efficient (randomized) approximation scheme.

1.1. Main results.

We focus on anti-ferromagnetic Potts models on *random regular bipartite graphs* at sufficiently low temperature. The probabilistic model of random regular bipartite graphs exhibits a special combinatorial structure. It can be shown that wiht high probability, the random regular bipartite graph is of good expansion ([JKP20, GGS21, LLLM22]). We will discuss details in Section 2.2.

Firstly we show the slow mixing of single-site Glauber dynamics. The Glauber dynamics P^{GD} is one of the most celebrated Markov chains with local updates. To sample from a target distribution π on Ω, starting from an initial state $X_0 \in \Omega$, at each step we update the current state X_t as following:

- pick a vertex $v \in V$ uniformly at random;
- update $X_{t+1} \sim \pi(\cdot \mid X_{t+1}(V \setminus \{v\}) = X_t(V \setminus \{v\}))$.

We show that it has an exponentially slow mixing time on the anti-ferromagnetic q-Potts models on random bipartite regular graphs at sufficiently low temperature.

Theorem 1.1. *There exist absolute constants $C_1, C_2 > 0$ such that, for every positive integers $q, \Delta \geq 3$ and a real number $\beta > 0$ satisfying $\Delta > C_1 q \ln q$ and $\beta > C_2 q^2/\Delta$, the following holds with high probability over G drawn from all random Δ-regular bipartite $2n$-vertex graphs for all sufficiently large positive integers $n \in \mathbb{N}$. The Glauber dynamics P^{GD} for the anti-ferromagnetic q-Potts models on G at inverse temperature β has a mixing time at least e^{Tn} for some factor $T = T(q, \Delta, \beta) > 0$.*

Despite the slow mixing time (Theorem 1.1), our second result shows that there exists an efficient deterministic algorithm to output an approximation to the partition function of the anti-ferromagnetic Potts models on random bipartite regular graphs at low temperature.

Theorem 1.2. *There exist a positive integer $q_0 \geq 3$ and a real number $C > 0$ such that for positive integers $q \geq q_0$, $\Delta \geq 4$ and a real number $\beta > 0$ satisfying that $\Delta(1-\mathrm{e}^{-\beta}) \geq C(q \ln(q\Delta/(1-\mathrm{e}^{-\beta})))^2$ and $\Delta\beta(1-\mathrm{e}^{-\beta})^2 \geq C(q \ln(q\Delta/(1-\mathrm{e}^{-\beta})))^2$, the following holds with high probability over G drawn uniformly from all Δ-regular bipartite $2n$-vertex graphs at random for every sufficiently large integer $n \in \mathbb{N}$. There exists a deterministic algorithm to output $\widehat{Z}$ such that $1 - \varepsilon \leq \widehat{Z}/Z_{G,q}(\beta) \leq 1 + \varepsilon$ in running time $(n/\varepsilon)^{\tau(\Delta)}$ for a computable function $\tau : \mathbb{R} \to (0, +\infty)$ depending only on Δ,* i.e.*, an* **FPTAS** *for counting the partition function of the anti-ferromagnetic q-Potts model on G at β.*

1.2. Technical overview.

We state here a brief overview of techniques used in our work, with some details in Sections 2 and 3.

Existence of ordered phases. To show Theorems 1.1 and 1.2, we investigate the geometric structure of the configuration space of the anti-ferromagnetic Potts model on random regular bipartite graphs. A key observation is that, at sufficiently low temperature, most of the configurations follow one of typical 'coloring patterns', similar to the anti-ferromagnetic 3-states case on the lattice $\mathbb{Z}^\Delta$ in [GKRS15, FS19]. Based on this kind of observation, we apply the concept of patterns in [JKP20][3] to give a description of ordered phases and provide some properties which play crucial roles in our proof.

Conductance. The conductance of Markov chains, introduced by Sinclair and Jerrum [SJ89], has been widely used to show both rapid and torpid mixing of a Markov chain. By a type of the Cheeger's inequality, there exists a bridge between the conductance of a given Markov chain and the second largest eigenvalue of the Laplacian matrix of it, leading to both upper and lower bound to its mixing time. For anti-ferromagnetic Ising models ($q = 2$) on a family of regular bipartite expander graphs, a recent work by Geisler et al. [GKSW26] has shown that the conductance of Glauber dynamics is exponentially small, and thus the mixing time is exponential in the size of the instance.

Like the torpid mixing of P^{GD} on the anti-ferromagnetic Ising case in [GKSW26], to show Theorem 1.1, we apply the method of conductance. We show how to find a subset of the state space with small conductance and apply a simpler lower bound for the mixing time by Levin and Peres [LP17] than the one induced by the Cheeger's inequality. To find the very subset, we also prove some useful results to describe the geometry of the anti-ferromagnetic q-Potts models on random regular bipartite graphs.

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[3] We note here that the concept of patterns in [JKP20] is a special case of the maximal bi-cliques in [GGS21].

Abstract polymer models. The *abstract polymer model* together with *the cluster expansion* is a useful tool to study statistical mechanics models. In the field of theoretic computer science, the abstract polymer model has been successfully used in the design of **FPTAS** or **FPRAS** to the hard-core models ([JKP20, CGG$^+$21, LLLM22]), ferromagnetic Potts models ([JKP20, CGG$^+$21, GGS22]) on (bipartite) expander graphs, general spin systems ([GGS21]) and proper colorings ([JKP20, LLLM22]).

We follow the methodology in [JKP20, LLLM22, GGS21]. Based on our description of ordered phases, we specify each configuration in Ω with patterns. Therefore, for each pattern, we design a suitable abstract polymer model and follow a routine way to obtain an efficient approximation scheme to the partition function by verifying the Kotecký-Preiss condition.

1.3. Comparison with related works.

The torpid mixing of Glauber dynamics has been recently shown on the anti-ferromagnetic Ising models with uniform external fields by [GKSW26] on bipartite expander graphs. Their result for the slow mixing time relies on an observation that the set of configurations which is 'balanced' in some way owns a exponentially small weight. In this work, we aim to show a similar property for the anti-ferromagnetic Potts models on random regular bipartite graphs which are 'good expanders' with high probability. The challenge arises from the difficulty to define an appropriate collection of 'balanced configurations' and upper bound their total weights. To overcome this, we investigate the geometry of the configuration space and show that the configurations are highly concentrated around some specific 'ground states'. Therefore, we are able to define a proper 'boundary' and show the small conductance of the Glauber dynamics.

For approximately counting the anti-ferromagnetic Potts models on random regular bipartite graphs, it has been shown in [JKP20] that the state space can be viewed as a union of configurations specified by different coloring 'patterns' representing assigning almostly disjoint colors to each side for proper vertex-colorings. For the soft-constrained models, Galanis et al. [GGS21] apply maximal bi-cliques which is a generalization of patterns and has been successfully used to design an efficient sampler on the ferromagnetic case in [CGG$^+$21]. As a result, they design an **FPRAS** for counting the anti-ferromagnetic Potts models on random regular bipartite graphs, with a strong constraint on the relation between the degree Δ and the number of colors q. To loose the constraint on Δ with respect to q, we consider the concept of patterns in [JKP20] to adapt anti-ferromagnetic Potts models and provide a *deterministic* algorithm to approximate the partition function.

1.4. Organization of this paper.

We state preliminaries including some notations, definitions and useful lemmas in Section 2. Then in Section 3, we show some properties of the configuration space of the anti-ferromagnetic Potts models on random regular bipartite graphs. The proof of slow mixing time (Theorem 1.1) is shown in Section 4 and we provide our **FPTAS** to the partition function in Section 5. Finally, in Section 6, we conclude our work and state some discussions.

2. PRELIMINARIES

2.1. Mathematical notations.

We list notations used in the paper here. We use e to denote the natural logarithm base and ln to denote the natural logarithm function. The binary entropy function $H : [0, 1] \to \mathbb{R}$ is defined as $H(x) = -x \ln x - (1 - x) \ln (1 - x)$ with convention $H(0) = H(1) = 0$. For a natural number n, we use $[n]$ to denote the set $\{1, 2, \ldots, n\}$. For two sets A, B, we use $A \sqcup B$ to denote the disjoint union of A and B.

A graph G consists of a finite vertex set V and an edge set $E \subseteq V \times V$. Given a graph $G = (V, E)$, we denote by $N_G(S)$ the neighborhood of S for every $S \subseteq V$, *i.e.*,

$$N_G(S) := \{v \in V \setminus S \mid \exists u \in S, (u, v) \in E\}$$

and for simplicity, use $N_G(v)$ to denote $N_G(\{v\})$ when $S = \{v\}$ for each $v \in V$. For two vertex subsets $S, T \subseteq V$, we use $E_G(S, T) := \{(u, v) \in E : \ u \in S \wedge v \in T\}$ to denote the edges between S

and T. For a subset $S \subseteq V$ and an integer $\ell \geq 1$, we say S is ℓ-connected if the induced subgraph $G^\ell[S]$ is connected. For simplicity, we use $G = (V = \mathcal{L} \sqcup \mathcal{R}, E)$ to denote a *bipartite graph*, where $(\mathcal{L}, \mathcal{R})$ forms a partition of the vertex set V, and the edge set $E \subseteq \mathcal{L} \times \mathcal{R}$.

For an assignment $\sigma : U \to [q]$ and a subset $S \subseteq U$, we use $\sigma(S)$ to denote the partial assignment of σ projected on S, *i.e.*, $\sigma|_S$ and use $\sigma(v)$ to denote $\sigma(\{v\})$ for simplicity when $S = \{v\}$. With little abuse of notation, we also use $\sigma(v)$ to denote the color assigned to v by σ. For a color subset $C \subseteq [q]$, we denote by $\sigma^{-1}(C)$ the subset $S \subseteq U$ assigned colors in C by σ, *i.e.*, $\sigma^{-1}(C) := \{v \in U \mid \sigma(v) \in C\}$.

For a state space $[q]^U$, we use $d_H(\cdot, \cdot)$ to denote the Hamming distance on it, *i.e.*,

$$\forall \sigma, \tau \in [q]^U, \quad d_H(\sigma, \tau) = \sum_{v \in U} \mathbb{1}\left[\sigma(u) \neq \sigma(v)\right].$$

We extend $d_H(\cdot, \cdot)$ to subsets of $[q]^U$ as

$$\forall \Lambda, \Pi \subseteq [q]^U, \quad d_H(\Lambda, \Pi) = \min_{\sigma \in \Lambda, \tau \in \Pi} d_H(\sigma, \tau).$$

2.2. **Random regular bipartite graphs.** Now we discuss here some combinatorial properties of random regular bipartite graphs. For positive integers $\Delta, n \in \mathbb{N}_+$, we use $\mathcal{G}^{\mathsf{bip}}_{n,\Delta}$ to denote the probabilistic model of random Δ-regular bipartite graphs of $2n$ vertices.

Another important probabilistic model is the *configuration model*. Let $G = (V = \mathcal{L} \sqcup \mathcal{R}, E) \sim \mathcal{G}^{\mathsf{CM}}_{n,\Delta}$ be a bipartite multi-graph generated by the following procedure:

- The vertex set $V = \mathcal{L} \sqcup \mathcal{R}$ is set by $\mathcal{L} = [n]$ and $\mathcal{R} = [n]$.
- To generate E, we sample Δ perfect matchings of the complete bipartite graph $K_{n,n} = (\mathcal{L} \sqcup \mathcal{R}, \mathcal{L} \times \mathcal{R})$ uniformly at random and for each perfect matching, we add edges in it to E.

The following proposition builds a connection between two probabilistic models.

Proposition 2.1 ([Wor99])**.** *For an event $\mathcal{E}$, if $\mathbb{P}_{G \sim \mathcal{G}^{\mathsf{CM}}_{n,\Delta}}(G \in \mathcal{E}) = o(1)$, then $\mathbb{P}_{G \sim \mathcal{G}^{\mathsf{bip}}_{n,\Delta}}(G \in \mathcal{E}) = o(1)$.*

Perhaps the most important combinatorial property of random regular bipartite graphs is the *expansion*. At first, we consider the edge expansion. The following lemma is a refined version of the expander mixing lemma in random regular bipartite graphs.

Lemma 2.2. *Given a positive integer $\Delta \geq 3$ and a real number $\delta \in (0, 1)$, there exists a constant $C > 0$ such that the following holds with high probability over $G \sim \mathcal{G}^{\mathsf{bip}}_{n,\Delta}$ for every $\kappa \in (0, 1)$ satisfying $\delta^2 \Delta \kappa > C(1 - \ln \kappa)$ and sufficiently large positive integer $n \in \mathbb{N}$. For every $S \subseteq \mathcal{L}$ and $T \subseteq \mathcal{R}$ with $|S|, |T| \geq \kappa n$, it holds that*

$$|E_G(S, T)| \geq (1 - \delta) \frac{\Delta \cdot |S| \cdot |T|}{n}. \tag{3}$$

Proof. We show the statement holds with high probability over $G \sim \mathcal{G}^{\mathsf{CM}}_{n,\Delta}$ and by Proposition 2.1, it holds with high probability over random regular bipartite graphs. Set $\mathcal{E}$ be the event that there exists a pair of subsets $(S \subseteq \mathcal{L}, T \subseteq \mathcal{R})$ with $|S|, |T| \geq \kappa n$ such that $|E_G(S, T)| < (1 - \delta)\frac{\Delta \cdot |S| \cdot |T|}{n}$. For $S \subseteq \mathcal{L}$ and $T \subseteq \mathcal{R}$, let X be the number of edges in $E_G(S, T)$. Then it holds that X is a hypergeometric random variable with $X \sim \mathrm{HyperGeo}(\Delta n, \Delta|T|, \Delta|S|)$. We calculate its expectation as $\mathbf{E}[X] = \frac{\Delta \cdot |S| \cdot |T|}{n}$. Using the Chernoff-Hoeffding inequality, it holds that

$$\mathbb{P}\left(X \leq (1 - \delta)\mathbf{E}[X]\right) \leq \exp\left(-\frac{\mathbf{E}[X]\,\delta^2}{2}\right) \leq \exp\left(-\frac{\delta^2 \Delta \cdot |S| \cdot |T|}{2n}\right).$$

By a union bound, it holds that

$$\begin{aligned}\mathbb{P}_{G\sim\mathcal{G}^{\mathsf{CM}}_{n,\Delta}}(G\sim\mathcal{E}) &\le \sum_{i=\kappa n}^{n}\sum_{j=\kappa n}^{n}\binom{n}{i}\binom{n}{j}\exp\left(-\frac{\delta^2\Delta\cdot|S|\cdot|T|}{2n}\right)\\ &\le \sum_{a,b\in\{\kappa,\kappa+1/n,\dots,1\}}\exp\left(n\left(H(a)+H(b)-\frac{\delta^2\Delta}{2}ab\right)\right)\\ &\le n^2\exp\left(-n\cdot\frac{4\ln n}{n}\right)\\ &\le n^{-2}\end{aligned}$$

if $\frac{\delta^2\Delta}{2}xy-H(x)-H(y)\ge\frac{4\ln n}{n}$ for every $x,y\in[\kappa,1]$. A simple bound for $(1-x)\ln(1-x)$ tells us that $H(x)\le x(1-\ln x)$ for every $x\ge\kappa$. Then it holds that there exists some absolute constant $C>0$ such that when $\delta^2\Delta\kappa>C\ln(e/\kappa)$ and sufficiently large $n\in\mathbb{N}$. Thus we conclude the lemma. ☐

The second kind of expansion is the *vertex expansion*. For two numbers $\rho\in(0,1)$ and $\alpha>0$ such that $\rho\alpha\le1$, we say a bipartite graph $G=(V=\mathcal{L}\sqcup\mathcal{R},E)$ is a *bipartite* (ρ,α)*-expander* if $|N_G(S)|\le\alpha|S|$ for every vertex subset $S\subseteq\mathcal{L}$ with $|S|\le\rho|\mathcal{L}|$ and $S\subseteq\mathcal{R}$ with $|S|\le\rho|\mathcal{R}|$. We denote by $\mathcal{G}^{\rho,\alpha}_{n,\Delta}$ the collection of all Δ-regular bipartite (ρ,α)-expanders of $2n$ vertices. We use the following lemma in [JKP20] that a random regular bipartite graph is a bipartite expander with high probability.

Lemma 2.3 (Lemma 23 in [JKP20]). *For sufficiently large Δ, it holds that with high probability over $G\sim\mathcal{G}^{\mathsf{bip}}_{n,\Delta}$, $G\in\mathcal{G}^{\rho,\alpha}_{n,\Delta}$ for $\rho=\frac{4\ln\Delta}{\Delta}$ and $\alpha=\frac{\Delta}{4\ln\Delta}-\frac{1}{2}$.*

We also need the following expansion derived by the spectral method in [GGS21].

Lemma 2.4 ([GGS21]). *For a Δ-regular bipartite graph $G=(V=\mathcal{L}\sqcup\mathcal{R},E)$ such that the second largest eigenvalue of G satisfies that $\lambda_2(G)\le\lambda$, then it holds that for every $S\subseteq\mathcal{L}$ and $S\subseteq\mathcal{R}$,*

$$|N_G(S)|\ge\frac{\Delta^2}{\lambda^2+(\Delta^2-\lambda^2)|S|/n}|S|.$$

The following vertex expansion is a direct corollary of Lemma 2.4.

Corollary 2.5. *For positive integers $\Delta\ge4$, the following holds with high probability over $G\sim\mathcal{G}^{\mathsf{bip}}_{n,\Delta}$. For every $\theta\in(0,1)$ and a vertex subset $S\subseteq V$ such that $|S|\le\theta n$, it holds that*

$$|N_G(S)|\ge\left(\frac{\Delta}{4+(\Delta-4)\theta}-1\right)|S|.$$

Proof. We use the result in [BDH22] to show that with high probability over $G=(V=\mathcal{L}\sqcup\mathcal{R},E)\sim\mathcal{G}^{\mathsf{bip}}_{n,\Delta}$, $\lambda_2(G)\le2\sqrt{\Delta}$. Then by Lemma 2.4, for every subset $A\subseteq\mathcal{L}$ and $A\subseteq\mathcal{R}$, it holds that

$$|N_G(A)|\ge\frac{\Delta^2}{4\Delta+(\Delta^2-4\Delta)|A|/n}|A|=\frac{\Delta}{4+(\Delta-4)|A|/n}|A|.$$

Then for every $S\subseteq V$ with $|S|\le\theta n$, it holds that

$$\begin{aligned}|N_G(S\cap\mathcal{L})|&\ge\frac{\Delta}{4+(\Delta-4)|S\cap\mathcal{L}|/n}|S\cap\mathcal{L}|\ge\frac{\Delta}{4+(\Delta-4)\theta}|S\cap\mathcal{L}|,\\ |N_G(S\cap\mathcal{R})|&\ge\frac{\Delta}{4+(\Delta-4)|S\cap\mathcal{R}|/n}|S\cap\mathcal{R}|\ge\frac{\Delta}{4+(\Delta-4)\theta}|S\cap\mathcal{R}|.\end{aligned}$$

Thus we obtain that

$$|N_G(S \cap \mathcal{L})| + |N_G(S \cap \mathcal{R})| \geq \frac{\Delta |S|}{4 + (\Delta - 4)\theta},$$

which concludes the lemma by noting that $N_G(S) = \big(N_G(S \cap \mathcal{L}) \cup N_G(S \cap \mathcal{R})\big) \setminus S$. □

2.3. **Markov chains.** We state here some basic definitions and lemmas for Markov chains and for more details, please see [LP17].

For a state space Ω, a Markov chain M consists of a transition kernel P on Ω with its stationary distribution π. For brevity, we sometimes use the transition matrix P to denote the Markov chain M. In this work, we only consider *ergodic* Markov chains (see details in [LP17]), and for an ergodic Markov chain M, we say it is reversible with respect to π if the detailed balanced equation holds:

$$\forall \sigma, \tau \in \Omega, \quad \pi(\sigma) P(\sigma, \tau) = \pi(\tau) P(\tau, \sigma). \tag{4}$$

Recall the Glauber dynamics P^{GD} with its stationary distribution π. It is direct to verify the following fact.

Fact 2.6. *The single-site Glauber dynamics P^{GD} is reversible w.r.t. its stationary distribution π.*

Proof. We verify (4) for P^{GD} and π and it is direct to show the fact. □

We put our attention on the *mixing time* of Markov chains. For two probability distributions μ, ν on Ω, the *total variation distance* is defined as

$$\|\mu - \nu\|_{\mathrm{TV}} := \frac{1}{2} \sum_{x \in \Omega} |\mu(x) - \nu(x)| = \max_{\Lambda \subseteq \Omega} |\mu(\Lambda) - \nu(\Lambda)|.$$

For a positive real $\varepsilon \in (0,1)$, define the *mixing time* of M with error ε as

$$\tau_{\mathrm{mix}}(M, \varepsilon) := \inf \left\{ t > 0 \,:\, \forall x \in \Omega, \left\| P^t(x, \cdot) - \pi \right\|_{\mathrm{TV}} \leq \varepsilon \right\}.$$

We refer to the value of $\tau_{\mathrm{mix}}(M, \cdot)$ when $\varepsilon = 1/4$ as the mixing time of M, *i.e.*,

$$\tau_{\mathrm{mix}}(M) := \tau_{\mathrm{mix}}(M, 1/4).$$

For $\sigma, \tau \in \Omega$, we define $Q(\sigma \to \tau) := \pi(\sigma) P(\sigma, \tau)$ and for two subsets $\Lambda, \Pi \subseteq \Omega$, we define

$$Q(\Lambda \to \Pi) := \sum_{\sigma \in S} \sum_{\tau \in T} Q(\sigma \to \tau).$$

The *conductance* of a subset $\Lambda \subseteq \Omega$ is defined as:

$$\Phi_M(\Lambda) := \frac{Q(\Lambda \to (\Omega \setminus \Lambda))}{\pi(\Lambda)}$$

and the conductance of M is defined by

$$\Phi_M := \inf_{\Lambda \subseteq \Omega : 0 < \pi(\Lambda) \leq 1/2} \Phi_M(\Lambda).$$

The following lemma shows that the mixing time can be lower bounded by the conductance.

Lemma 2.7 ([LP17, Theorem 7.4])**.** *For an ergodic Markov chain M, it holds that*

$$\tau_{\mathrm{mix}}(M) \geq \frac{1}{4\Phi_M}.$$

2.4. **Abstract polymer model.** In this part, we introduce the *abstract polymer model*. The abstract polymer model $\mathcal{M} = (\mathcal{C}, \sim)$ consists of the collection $\mathcal{C}$ of *abstract polymers* and a binary relation $\sim \subseteq \mathcal{C} \times \mathcal{C}$. A polymer $\gamma = (V_\gamma, w_\gamma)$ is a subset $V_\gamma \subseteq V$ equipped with a weight function w_γ. With abuse of notation, we sometimes use γ to denote V_γ and set the size $|\gamma| := |V_\gamma|$. For two polymers γ_1, γ_2, we say γ_1 is *compatible* with γ_2 if $\gamma_1 \not\sim \gamma_2$ and otherwise they are *incompatible*. We denote by $\mathcal{K}$ the collection of finite subsets Γ of $\mathcal{C}$ such that polymers in Γ are mutually compatible. Then we define the partition function of the abstract polymer model $\mathcal{M}$ as

$$\Xi(\mathcal{M}) := \sum_{\Gamma \in \mathcal{K}} \prod_{\gamma \in \Gamma} w_\gamma.$$

Example: abstract polymer models induced by a graph. Mostly we consider the abstract polymer model induced by a graph G. Then we would define a polymer $\gamma = (V_\gamma, w_\gamma)$ via a connected subgraph V_γ. We say two polymers $\gamma_1 \sim \gamma_2$ if $V_{\gamma_1} \cup V_{\gamma_2}$ forms a connected subgraph in G.

The famous Kotecký-Preiss condition in [KP86] is of great significance to compute Ξ.

Condition 2.8 (Kotecký-Preiss condition)**.** *We say* $(\mathcal{C}, \sim)$ *satisfies Kotecký-Preiss condition if there exists a function* $g : \mathcal{C} \to \mathbb{R}_{\geq 0}$ *such that for every* $\gamma \in \Gamma$,

$$\sum_{\gamma' \sim \gamma} |w_{\gamma'}| \mathrm{e}^{|\gamma'| + g(\gamma')} \leq g(\gamma).$$

We use the following theorem to give an **FPTAS** to Ξ together with Condition 2.8.

Theorem 2.9 (Theorem 8 in [JKP20])**.** *Fix* $\Delta > 0$ *and a graph family* $\mathcal{G}$ *of maximum degree at most* Δ. *Suppose that the followings hold for the polymer models induced by* $\mathcal{G}$ *and a function* $g(\cdot)$*:*

1. *There exist two constants* $C_1, C_2 > 0$ *such that for every connected subgraph* γ*, it takes at most* $|\gamma|^{C_1} \mathrm{e}^{C_2 |\gamma|}$ *time to determine whether* γ *is in the model, and then compute* w_γ *and* $g(\gamma)$*.*
2. *There exists some constants* $C = C(\Delta) > 0$ *such that for every* $G \in \mathcal{G}$ *and each polymer* γ *induced by* G*,* $g(\gamma) \geq C|\gamma|$*.*
3. *Condition 2.8 holds with choice* g*.*

Then there exists an **FPTAS** *to approximate the partition function* Ξ *of the abstract polymer model induced by every graph* $G \in \mathcal{G}$ *with running time* $O\left((n/\varepsilon)^{\tau(\Delta)}\right)$ *for* $\tau(\Delta) = (\ln \Delta + C_2)/C(\Delta)$*.*

3. ANTI-FERROMAGNETIC POTTS MODELS ON RANDOM REGULAR BIPARTITE GRAPHS

In this section, we investigate the geometry of the configuration space of the anti-ferromagnetic q-Potts model. As mentioned above, we aim to show that, at sufficiently low temperature, the anti-ferromagnetic q-state Potts models on $\mathcal{G}^{\mathsf{bip}}_{n,\Delta}$ follows some ordered phases with high probability. Briefly, for each configuration in Ω, there exists a coloring pattern such that it assigns almost disjoint colors to each side, and most of vertices agree with the pattern under the configuration. To describe ordered phases, we formalize the definition of *patterns* as [JKP20].

Definition 3.1 (Patterns)**.** For a color set $[q] = \{1, \ldots, q\}$ and two subsets $A, B \subseteq [q]$, we say (A, B) is a *pattern* if A and B form a partition of $[q]$, *i.e.*, $[q] = A \sqcup B$ and $|A| \neq 0$, $|B| \neq 0$. We denote by $\mathscr{P}$ the collection of all patterns.

Recall that in [JKP20], we say a configuration $\sigma \in \Omega$ *agrees* with (A, B) at $v \in V$ if $\sigma(v) \in A$ when $v \in \mathcal{L}$ or $\sigma(v) \in B$ when $v \in \mathcal{R}$; otherwise we say σ *disagrees* with it at v. And given a pattern (A, B) and a vertex subset $S \subseteq V$, define $\chi_{A,B}(S)$ as the collection of configurations $\sigma \in \Omega$ such that σ agrees with (A, B) at for every $v \in V \setminus S$ and disagrees with the pattern at $v \in V$. We remark here that unlike the case in [JKP20], we allow that the configuration is not necessarily a proper coloring.

To capture the geometric properties of the configuration space Ω, define the following rank function $\mathsf{rank} : \Omega \to \mathbb{N}$ by

$$\forall \sigma \in \Omega, \quad \mathsf{rank}(\sigma) = \inf \{\ell \in \mathbb{N} : \exists (A, B) \in \mathscr{P}, S \subseteq V, |S| = \ell \wedge \sigma \in \chi_{A,B}(S)\}.$$

Furthermore, we separate Ω according to the ranks rank as:

$$\forall \ell \in \mathbb{N}, \quad \Omega^{\ell} := \{\sigma \in \Omega : \mathsf{rank}(\sigma) = \ell\}$$

and define $\Omega^{\leq \ell} := \cup_{t \leq \ell} \Omega^t, \Omega^{>\ell} := \cup_{t>\ell} \Omega^t$ and the partition functions $Z^{\leq \ell} := \sum_{\sigma \in \Omega^{\leq \ell}} w(\sigma), Z^{>\ell} := \sum_{\sigma \in \Omega^{>\ell}} w(\sigma)$ restricted on them respectively. For brevity, given a parameter $\theta \in (0, 1)$, a parameter $\theta \in (0, 1)$, we say a configuration σ is of a low rank if $\mathsf{rank}(\sigma) \leq \theta n$ and otherwise it has a high rank. Moreover, for a configuration $\sigma \in \chi_{A,B}(S)$, we say σ is *specified by* (A, B) *at rank* $|S|$.

We define the total weight of $\chi_{A,B}(S)$ by

$$W_{A,B}(S) := \sum_{\sigma \in \chi_{A,B}(S)} w(\sigma).$$

The following lemma gives an upper bound to $W_{A,B}(S)$.

Lemma 3.2. *For every pattern* $(A, B) \in \mathscr{P}$ *and a vertex subset* $S \subseteq V$*, it holds that*

$$W_{A,B}(S) \leq |A|^n |B|^n \left(\frac{|A|}{|B|}\right)^{|S\cap\mathcal{R}|-|S\cap\mathcal{L}|} \left(1 - \frac{1 - \mathrm{e}^{-\beta}}{|A|}\right)^{|N_G(S)\cap\mathcal{L}|} \left(1 - \frac{1 - \mathrm{e}^{-\beta}}{|B|}\right)^{|N_G(S)\cap\mathcal{R}|}.$$

Proof. The proof is similar to [JKP20, Lemma 27] with some modifications to adapt the Potts model. We calculate $W_{A,B}(S)$ by the following steps using the multiplication rule.

- At first we consider all partial colorings on S (there are at most $|A|^{|S\cap\mathcal{R}|}|B|^{|S\cap\mathcal{L}|}$ partial colorings of weight 1);
- secondly we enumerate the vertices $T \subseteq N_G(S)$ with the same colors as their neighbors in S (they contribute $\mathrm{e}^{-\beta|E_G(S,T)|} \leq \mathrm{e}^{-\beta|T|}$);
- then we consider each vertex $v \in N_G(S) \setminus T$ (there are at most $(|A| - 1)^{|(N_G(S)\cap\mathcal{L})\setminus T|}(|B| - 1)^{|(N_G(S)\cap\mathcal{R})\setminus T|}$ partial colorings of weight 1);
- lastly we calculate the number of colorings on $V \setminus (S \cup N_G(S))$; there are
$$|A|^{n-|S\cap\mathcal{L}|-|N_G(S)\cap\mathcal{L}|}|B|^{n-|S\cap\mathcal{R}|-|N_G(S)\cap\mathcal{R}|}$$
partial colorings contributing 1.

We combine all steps and arrange it to obtain that

$$\begin{aligned} W_{A,B}(S) &\leq |A|^n |B|^n \left(\frac{|A|}{|B|}\right)^{|S\cap\mathcal{R}|-|S\cap\mathcal{L}|} \left(\frac{\mathrm{e}^{-\beta} + |A| - 1}{|A|}\right)^{|N_G(S)\cap\mathcal{L}|} \left(\frac{\mathrm{e}^{-\beta} + |B| - 1}{|B|}\right)^{|N_G(S)\cap\mathcal{R}|} \\ &= |A|^n |B|^n \left(\frac{|A|}{|B|}\right)^{|S\cap\mathcal{R}|-|S\cap\mathcal{L}|} \left(1 - \frac{1 - \mathrm{e}^{-\beta}}{|A|}\right)^{|N_G(S)\cap\mathcal{L}|} \left(1 - \frac{1 - \mathrm{e}^{-\beta}}{|B|}\right)^{|N_G(S)\cap\mathcal{R}|}. \end{aligned}$$

Thus we conclude the lemma. □

3.1. **Configurations at low ranks.**

In this part, we present some properties of configurations at low ranks. For a subset $\Lambda \subseteq \Omega$, define the *out-boundary* of Λ by

$$\partial_{\mathrm{out}}\Lambda := \{\sigma \in \Omega \setminus \Lambda : d_H(\sigma, \Lambda) = 1\}.$$

We consider the total weight in the out-boundary of configurations of low ranks specified by a given pattern. Given a pattern $(A, B) \in \mathscr{P}$, define the subspace $\Omega^{\leq \theta n}_{A,B}$ as

$$\Omega^{\leq \theta n}_{A,B} := \bigcup_{S \subseteq V : |S| \leq \theta n} \chi_{A,B}(S).$$

Lemma 3.3. *For every pattern* $(A,B) \in \mathscr{P}$ *and a real number* $\theta \in (0,1)$*, it holds that*

$$\sum_{\sigma \in \partial_{\mathrm{out}} \Omega^{\le \theta n}_{A,B}} w(\sigma) \le Z^{>\theta n} + 2^q \binom{2n}{\theta n} \binom{2n}{\theta n + 1} \left(\frac{q-1}{2}\right)^{2n} q^{2\theta n + 1}.$$

The following proposition about the interaction between two different patterns is useful in the proof of Lemma 3.3.

Proposition 3.4. *For two different patterns* $(A,B),(C,D) \in \mathscr{P}$ *and subsets* $S,T \subseteq V$*, it holds that*

$$\sum_{\sigma \in \chi_{A,B}(S) \cap \chi_{C,D}(T)} w(\sigma) \le \left(\frac{q-1}{2}\right)^{2n} q^{|S|+|T|}.$$

Proof. The proof is implicitly stated in [JKP20, GGS21] and we present here for completeness. Assume that A is of the largest size of A,B,C,D. We enumerate $\sigma \in \chi_{A,B}(S) \cap \chi_{C,D}(T)$ in the following way:

- firstly we color the subset of $\mathcal{L}$ agreeing with both A and C;
- then we color the subset of $\mathcal{R}$ agreeing with B and D;
- lastly we color the remaining vertices which is a subset of $S \cup T$.

By (1) and (1), it holds that $w(\sigma) \le 1$ for every $\sigma \in \Omega$. Then we obtain that

$$\begin{aligned}\sum_{\sigma \in \chi_{A,B}(S) \cap \chi_{C,D}(T)} w(\sigma) &\le |\chi_{A,B}(S) \cap \chi_{C,D}(T)| \\ &\le (|A|-1)^n |B|^n q^{|S|+|T|} \\ &\le \left(\frac{|A|-1+|B|}{2}\right)^{2n} q^{|S|+|T|} \\ &= \left(\frac{q-1}{2}\right)^{2n} q^{|S|+|T|}.\end{aligned}$$

Then we conclude the proposition. □

Proof of Lemma 3.3. Let

$$\Lambda := \bigcup_{S \in \binom{V}{\theta n + 1}} \chi_{A,B}(S)$$

be the collection of all configurations specified by (A,B) at rank $\theta n+1$. We claim that $\partial_{\mathrm{out}} \Omega^{\le \theta n}_{A,B} \subseteq \Lambda$. By definition, for every $\sigma \in \partial_{\mathrm{out}} \Omega^{\le \theta n}_{A,B}$, there exists $\tau \in \chi_{A,B}(S)$ such that $|S| \le \theta n$ and $d_H(\sigma,\tau) = 1$. Suppose that σ and τ differ at the vertex v. A simple argument shows that $v \notin S$ and σ disagrees with (A,B) at v, otherwise σ must be in $\chi_{A,B}(S)$ which leads to a contradiction. Then it holds that $\sigma \in \chi_{A,B}(S \cup \{v\})$. It is trivial to see $|S| = \theta n$ (otherwise $\sigma \in \Omega^{\le \theta n}_{A,B}$). Then we obtain that $\sigma \in \Lambda$.

Then we bound $\sum_{\sigma \in \Lambda} w(\sigma)$. We partition Λ according to ranks and calculate that

$$\begin{aligned}\sum_{\sigma \in \Lambda} w(\sigma) &= \sum_{\sigma \in \Lambda : \mathsf{rank}(\sigma) \le \theta n} w(\sigma) + \sum_{\sigma \in \Lambda : \mathsf{rank}(\sigma) > \theta n} w(\sigma) \\ &\le \sum_{(C,D) \ne (A,B) \in \mathscr{P}} \sum_{S \in \binom{V}{\theta n + 1}} \sum_{T \in \binom{V}{\theta n}} |\chi_{A,B}(S) \cap \chi_{C,D}(T)| + Z^{>\theta n}\end{aligned}$$

$$\leq Z^{>\theta n} + 2^q \binom{2n}{\theta n}\binom{2n}{\theta n+1}\left(\frac{q-1}{2}\right)^{2n} q^{2\theta n+1}$$

by Proposition 3.4. □

3.2. **Tail bounds at high ranks.** In this part, we show a 'tail bound' at high ranks. At first, the following claim which is implicit stated in [JKP20, Lemma 28] and [GGS21, Lemma 10] shows that if $\sigma \in \Omega^{>\ell}$ for a large ℓ, then there exists a color assigned to many vertices in both sides.

Claim 3.5. *For $\theta \in (0,1)$ and every $\sigma \in \Omega^{>\theta n}$, there exists a color $i \in [q]$ such that $\left|\sigma^{-1}(\{i\}) \cap \mathcal{L}\right| > \theta n/q$ and $\left|\sigma^{-1}(\{i\}) \cap \mathcal{R}\right| > \theta n/q$.*

Proof. Assume that for every $i \in [q]$, one of the followings must hold that $\left|\sigma^{-1}(\{i\}) \cap \mathcal{L}\right| \leq \theta n/q$ or $\left|\sigma^{-1}(\{i\}) \cap \mathcal{R}\right| \leq \theta n/q$. Define

$$\begin{aligned} A' &:= \left\{i \in [q] : \ \left|\sigma^{-1}(\{i\}) \cap \mathcal{L}\right| > \theta n/q\right\}, \\ B' &:= \left\{i \in [q] : \ \left|\sigma^{-1}(\{i\}) \cap \mathcal{R}\right| > \theta n/q\right\}. \end{aligned}$$

Then by assumption $A' \cap B' = \emptyset$. Pick a pattern $(A,B) \in \mathscr{P}$ such that $A' \subseteq A$ and $B' \subseteq B$.

Set $S = (\mathcal{L} \backslash \sigma^{-1}(A)) \cup (R \backslash \sigma^{-1}(B))$ and thus $\sigma \in \chi_{A,B}(S)$. By assumption, for every color $i \in A'$, $\left|\sigma^{-1}(\{i\}) \cap \mathcal{R}\right| \leq \theta n/q$, for every color $i \in B'$, $\left|\sigma^{-1}(\{i\}) \cap \mathcal{L}\right| \leq \theta n/q$ and for $i \in [q] \setminus (A' \cup B')$, $\left|\sigma^{-1}(\{i\}) \cap T\right| \leq \theta n/q$ for both $T = \mathcal{L}$ and $T = \mathcal{R}$. Then we bound the size of S by

$$\begin{aligned} |S| &= \sum_{i \in A} \left|\sigma^{-1}(\{i\}) \cap \mathcal{R}\right| + \sum_{i \in B} \left|\sigma^{-1}(\{i\}) \cap \mathcal{L}\right| \\ &\leq (\theta n/q)\left|A'\right| + (\theta n/q)\left|B'\right| + (\theta n/q)(q - \left|A'\right| - \left|B'\right|) \\ &= \theta n. \end{aligned}$$

This means that $\sigma \in \Omega^{\leq \theta n}$, leading to a contradiction. Then we conclude the claim. □

Lemma 3.6. *Given positive integers $q, \Delta \geq 3$, there exists some constant $C > 0$ such that for real numbers $\delta, \theta \in (0,1), \beta > 0$ satisfying $\delta^2 \Delta(\theta/q) > C(1 - \ln(\theta/q))$ and*

$$\beta(1-\delta)\Delta(\theta/q)^2 + 2(\theta/q)\ln q - 2\ln 2 - 2 \cdot H(\theta/q) \geq 2/q,$$

the following holds with probability $1 - o_n(1)$ over $G \sim \mathcal{G}^{\mathsf{bip}}_{n,\Delta}$ for a sufficiently large positive integer $n \in \mathbb{N}$:

$$\mu_{G,q;\beta}\left(\Omega^{>\theta n}\right) \leq \mathrm{e}^{-n/q}.$$

Proof. For a random Δ-regular bipartite graph $G = (V = \mathcal{L} \sqcup \mathcal{R}, E) \sim \mathcal{G}^{\mathsf{bip}}_{n,\Delta}$, by Lemma 2.2, with high probability the lower bound (3) holds for every $S \subseteq \mathcal{L}$ and $T \subseteq \mathcal{R}$ of size at least $(\theta/q)n$ when $\delta^2\Delta(\theta/q) \geq C(1 - \ln(\theta/q))$ and the term n is sufficiently large. Then we assume that the statement in Lemma 2.2 holds.

For every color $i \in [q]$, define Ω_i^{big} as the collection of configurations σ satisfying $\left|\sigma^{-1}(\{i\}) \cap \mathcal{L}\right| > \theta n/q$ and $\left|\sigma^{-1}(\{i\}) \cap \mathcal{R}\right| > \theta n/q$. By Claim 3.5, it holds that $\Omega^{>\theta n} \subseteq \cup_{i \in [q]} \Omega_i^{\mathsf{big}}$. Then we upper bound $\mu(\Omega^{>\theta n})$ through $\mu(\cup_{i\in[q]}\Omega_i^{\mathsf{big}})$ as

$$\begin{aligned} \mu\left(\Omega^{>\theta n}\right) &\leq \mu\left(\cup_{i \in [q]} \Omega_i^{\mathsf{big}}\right) \\ &\leq \sum_{i \in [q]} \mu\left(\Omega_i^{\mathsf{big}}\right) \end{aligned}$$

$$
\begin{aligned}
&\le \frac{\sum_{i\in[q]}\left|\Omega_i^{\mathsf{big}}\right|\exp\left(-\beta(1-\delta)\Delta(\theta/q)^2n\right)}{Z_{G,q}(\beta)}\\
&\le q\binom{n}{(\theta/q)n}^2 q^{2n-2(\theta/q)n}\frac{\exp\left(-\beta(1-\delta)\Delta(\theta/q)^2n\right)}{(q/2)^{2n}}\\
&\le q\exp\left(2nH(\theta/q)-2(\theta/q)n\ln q-\beta(1-\delta)\Delta(\theta/q)^2n+2n\ln 2\right)\\
&\le \mathrm{e}^{-n/q}
\end{aligned}
$$

where we use the inequality that $\binom{n}{k}\le\exp(nH(k/n))$ and a crude lower bound for the partition function $Z_{G,q}(\beta)\ge(q/2)^{2n}$ (by considering all proper colorings). Thus we conclude the lemma. $\square$

4. SLOW MIXING OF GLAUBER DYNAMICS

In this section, we show how to prove the torpid mixing of the single-site Glauber dynamics P^{GD} for anti-ferromagnetic q-Potts models on random regular bipartite graphs $\mathcal{G}^{\mathsf{bip}}_{n,\Delta}$ (*i.e.*, Theorem 1.1). Based on properties in Section 3, the key ingredient is to pick an appropriate subset of Ω and to show that is has an exponentially small conductance.

Pick a pattern $(A,B)\in\mathscr{P}$ such that $A=\{1,\dots,\lceil q/2\rceil\}$ and $B=\{\lceil q/2\rceil+1,\dots,q\}$. Recall the definition of $\Omega^{\le\theta n}_{A,B}$, the collection of configurations specified by (A,B) at low ranks:

$$\Omega^{\le\theta n}_{A,B}=\bigcup_{S\subseteq V:|S|\le\theta n}\chi_{A,B}(S).$$

We show that with high probability over random Δ-regular bipartite graphs, $\Omega^{\le\theta n}_{A,B}$ has a small conductance.

Lemma 4.1. *Given positive integers $q,\Delta\ge3$, there exists some constant $C>0$ such that for real numbers $\delta,\theta\in(0,1)$, $\beta>0$ satisfying $\delta^2\Delta(\theta/q)>C(1-\ln(\theta/q))$ and*

$$\beta(1-\delta)\Delta(\theta/q)^2-2H(\theta/q)-2(1-\theta/q)\ln q+\ln\left(\lceil q/2\rceil\lfloor q/2\rfloor\right)>2/q,$$

the following holds with probability $1-o_n(1)$ over $G\sim\mathcal{G}^{\mathsf{bip}}_{n,\Delta}$ for a sufficiently large positive integer $n\in\mathbb{N}$. There exists a factor $T=T(q,\theta,\beta)>0$ such that

$$\Phi_{P^{\mathsf{GD}}}\left(\Omega^{\le\theta n}_{A,B}\right)\le\mathrm{e}^{-Tn}.$$

Proof. By Lemma 2.2, there exists some constant $C>0$ such that when $\delta^2\Delta(\theta/q)>C(1-\ln(\theta/q))$ and the term n is sufficiently large, with probability $1-o_n(1)$ over $G\sim\mathcal{G}^{\mathsf{bip}}_{n,\Delta}$, the lower bound (3) holds for every $S\subseteq\mathcal{L}$ and $T\subseteq\mathcal{R}$ with $|S|,|T|\ge(\theta/q)n$. Then we assume that such statement holds.

Since P^{GD} is reversible with respect to $\mu_{G,q;\beta}$ by Fact 2.6, by definition of $Q(\cdot\to\cdot)$ and the detailed balanced equation (4), it holds that

$$
\begin{aligned}
Q\left(\Omega^{\le\theta n}_{A,B}\to\left(\Omega\setminus\Omega^{\le\theta n}_{A,B}\right)\right)&=\sum_{\sigma\in\Omega^{\le\theta n}_{A,B}}\sum_{\tau\in\Omega\setminus\Omega^{\le\theta n}_{A,B}}\mu(\sigma)P^{\mathsf{GD}}(\sigma,\tau)\\
&=\sum_{\sigma\in\Omega^{\le\theta n}_{A,B}}\sum_{\tau\in\partial_{\mathrm{out}}\Omega^{\le\theta n}_{A,B}}\mu(\sigma)P^{\mathsf{GD}}(\sigma,\tau)\\
&=\sum_{\sigma\in\Omega^{\le\theta n}_{A,B}}\sum_{\tau\in\partial_{\mathrm{out}}\Omega^{\le\theta n}_{A,B}}\mu(\tau)P^{\mathsf{GD}}(\tau,\sigma)
\end{aligned}
$$

$$\le \sum_{\tau \in \partial_{\mathrm{out}} \Omega_{A,B}^{\le \theta n}} \mu(\tau)$$
$$= \mu\left(\partial_{\mathrm{out}} \Omega_{A,B}^{\le \theta n}\right).$$

Then we upper bound the conductance

$$\Phi_{P^{\mathsf{GD}}}\left(\Omega_{A,B}^{\le \theta n}\right) = \frac{Q\left(\Omega_{A,B}^{\le \theta n} \to \left(\Omega \setminus \Omega_{A,B}^{\le \theta n}\right)\right)}{\mu\left(\Omega_{A,B}^{\le \theta n}\right)} \le \frac{\mu\left(\partial_{\mathrm{out}} \Omega_{A,B}^{\le \theta n}\right)}{\mu(\Omega_{A,B}^{\le \theta n})} = \frac{w\left(\partial_{\mathrm{out}} \Omega_{A,B}^{\le \theta n}\right)}{w\left(\Omega_{A,B}^{\le \theta n}\right)}.$$

By Lemma 3.3 and a similar calculation in Lemma 3.6, it holds that

$$\begin{aligned} w\left(\partial_{\mathrm{out}} \Omega_{A,B}^{\le \theta n}\right) &\le Z^{>\theta n} + 2^q \binom{2n}{\theta n}\binom{2n}{\theta n + 1}\left(\frac{q-1}{2}\right)^{2n} q^{2\theta n+1} \\ &\le q\binom{n}{(\theta/q)n}^2 q^{2n-2(\theta/q)n} \mathrm{e}^{-\beta(1-\delta)\Delta(\theta/q)^2 n} + 2^q \binom{2n}{\theta n}\binom{2n}{\theta n + 1}\left(\frac{q-1}{2}\right)^{2n} q^{2\theta n+1} \\ &\le \mathrm{e}^{-T(q,\theta,\beta)n} (\lceil q/2 \rceil \lfloor q/2 \rfloor)^n \\ &\le \mathrm{e}^{-T(q,\theta,\beta)n} w\left(\Omega_{A,B}^{\le \theta n}\right) \end{aligned}$$

for some factor $T = T(q, \theta, \beta)$ and sufficiently large $n \in \mathbb{N}$ by using the crude bound $w\left(\Omega_{A,B}^{\le \theta n}\right) \ge (\lceil q/2 \rceil \lfloor q/2 \rfloor)^n$. Thus we conclude the lemma. ☐

Proof of Theorem 1.1. We pick parameters δ, θ and β to satisfy conditions in Lemma 4.1 and thus by Lemma 2.7, P^{GD} exhibits an exponential mixing time.

Pick $\delta = 1/100$ and $\theta = 1/50$. Then there exist absolute constants $C_1, C_2 > 0$ such that for every $\Delta > C_1 q \ln q$ and $\beta > C_2 q^2/\Delta$, the conditions in Lemma 4.1 holds and $\mu\left(\Omega_{A,B}^{\le \theta n}\right) < 1/2$ by symmetry for $A = \{1, \dots, \lceil q/2 \rceil\}$ and $B = \{\lceil q/2 \rceil + 1, \dots, q\}$. Then by Lemma 4.1, for all sufficiently large $n \in \mathbb{N}$, with high probability $1 - o_n(1)$ over $G \sim \mathcal{G}_{n,\Delta}^{\mathsf{bip}}$ it holds that $\Phi_{P^{\mathsf{GD}}}(\Omega_{A,B}^{\le \theta n}) \le \mathrm{e}^{-Tn}$ for some factor $T = T(q, \Delta, \beta)$. By Lemma 2.7, we conclude the theorem. ☐

5. APPROXIMATELY COUNTING ANTI-FERROMAGNETIC POTTS MODELS

In this section, we design an **FPTAS** for counting the partition function of anti-ferromagnetic Potts models on random bipartite regular graphs (Theorem 1.2). We apply the following ideas to approximate $Z_{G,q}(\beta)$ when G is a bipartite expander, which is an extension of the algorithm to count the number of proper colorings in [JKP20]:

1. firstly, we know that the major weight of $Z_{G,q}(\beta)$ lies in the configurations of rank at most θn by the tail bound Lemma 3.6;
2. then, we show that $Z^{\le \theta n}$ can be approximately estimated by summing up configurations specified by each pattern respectively;
3. finally, for each pattern, we design a proper abstract polymer model to estimate the weight of configurations specified by it.

To formalize the proof idea, we introduce the sparse subset used in [JKP20].

Definition 5.1 (Sparse subset [JKP20])**.** Given a graph $G = (V, E)$, for a vertex subset $S \subseteq V$, we say S is *sparse* if each 3-connect component in $G[S]$ is of size at most θn. We denote by $\Gamma^{\mathsf{sp}} = \Gamma^{\mathsf{sp}}(G, \theta)$ be the collection of all sparse subsets of G. Moreover, for a pattern $(A, B) \in \mathscr{P}$, we define $Z_{A,B}^{\mathsf{sp}} = \sum_{S \in \Gamma^{\mathsf{sp}}} W_{A,B}(S)$ as the partial partition function projected to sparse subsets with respect to (A, B).

The following lemmas are key ingredients to show Theorem 1.2.

Lemma 5.2. *Given positive integers* $q, \Delta \geq 3$ *and a real number* $\beta > 0$ *satisfying that* $4H(\theta/2) + 2\theta \ln q \leq 1/q$, *the following holds for every bipartite* $2n$-*vertex graph* $G = (V = \mathcal{L} \cup \mathcal{R}, E)$ *for sufficiently large integer* $n \in \mathbb{N}$:

$$1 \leq \frac{\sum_{(A,B)\in\mathscr{P}} \sum_{S\subseteq V:|S|\leq\theta n} W_{A,B}(S)}{Z^{\leq\theta n}} \leq 1 + \mathrm{e}^{-n/(100q)}.$$

Lemma 5.3. *There exist absolute constants* $C_0 \in (0,1)$ *and* $C_1, C_2 > 0$ *such that, for positive integers* $q \geq 3$ *and* $\Delta \geq 4$ *sufficiently large, and real numbers* $\beta > 0, \theta \in (0,1)$ *satisfying that* $\theta \geq 8\ln\Delta/\Delta$, $\theta < C_0$ *and*

$$2H(3\theta/2) + 3\theta \ln q \leq \frac{C_1(1-\mathrm{e}^{-\beta})}{q},$$

the following holds for every bipartite $2n$-*vertex graph* $G = (V = \mathcal{L} \cup \mathcal{R}, E)$ *for sufficiently large integer* $n \in \mathbb{N}$:

$$1 \leq \frac{\sum_{(A,B)\in\mathscr{P}} Z^{\mathsf{sp}}_{A,B}}{\sum_{(A,B)\in\mathscr{P}} \sum_{S\subseteq V:|S|\leq\theta n} W_{A,B}(S)} \leq 1 + \exp\left(-\frac{C_2(1-\mathrm{e}^{-\beta})}{q} n\right). \tag{5}$$

Lemma 5.4. *Given positive integers* $q, \Delta \geq 4$ *and real numbers* $\varepsilon, \theta \in (0,1)$, $\beta > 0$ *satisfying*

$$\frac{1-\mathrm{e}^{-\beta}}{q}\left(\frac{\Delta}{4+(\Delta-4)\theta} - 1\right) - \ln q \geq 5 + 9\ln\Delta,$$

the following holds with high probability over $G \sim \mathcal{G}^{\mathsf{bip}}_{n,\Delta}$. *For each pattern* $(A,B) \in \mathscr{P}$, *there exists a deterministic algorithm to output an* ε-*approximation to the quantity* $Z^{\mathsf{sp}}_{A,B}$ *running in time* $(n/\varepsilon)^{\tau(\Delta)}$ *for a computable function* $\tau : \mathbb{R} \to (0, +\infty)$ *depending only on* Δ.

We defer the proof of Lemma 5.2 in Section 5.1, the proof of Lemma 5.3 in Section 5.2 and the proof of Lemma 5.4 in Section 5.3.

Proof of Theorem 1.2. Pick parameters $\delta = \frac{1}{100}$ and $\theta = \frac{1-\mathrm{e}^{-\beta}}{10q\ln\frac{q\Delta}{1-\mathrm{e}^{-\beta}}}$. Then there exists an absolute constant $C > 0$ such that when $\Delta(1-\mathrm{e}^{-\beta}) \geq Cq^2\ln^2\frac{q\Delta}{1-\mathrm{e}^{-\beta}}$ and $\beta(1-\mathrm{e}^{-\beta})^2 \geq \frac{Cq^4\ln^2\frac{q\Delta}{1-\mathrm{e}^{-\beta}}}{\Delta}$, conditions in Lemmas 3.6 and 5.2 to 5.4 hold for sufficiently large positive integer $n \in \mathbb{N}$ and the bound (5) holds for $C_2 = C$.

When $\varepsilon \leq 5\mathrm{e}^{-C\frac{1-\mathrm{e}^{-\beta}}{q}n}$, we compute $Z_{G,q}(\beta)$ by a brute force enumeration of all configurations and it is direct to see that the algorithm terminates in time $\mathrm{poly}\left(n, \varepsilon^{-1}\right)$.

When $\varepsilon > 5\mathrm{e}^{-C\frac{1-\mathrm{e}^{-\beta}}{q}n}$, we have the following holds:

(1) $Z^{\leq\theta n}$ is an $(\varepsilon/5)$-approximation to $Z_{G,q}(\beta)$ by Lemma 3.6;

(2) the quantity

$$\sum_{(A,B)\in\mathscr{P}} \sum_{S\subseteq V:|S|\leq\theta n} W_{A,B}(S)$$

is an $(\varepsilon/5)$-approximation to $Z^{\leq\theta n}$ by Lemma 5.2;

(3) $\sum_{(A,B)\in\mathscr{P}} Z^{\mathsf{sp}}_{A,B}$ is an $(\varepsilon/5)$-approximation to the quantity $\sum_{(A,B)\in\mathscr{P}} \sum_{S\subseteq V:|S|\leq\theta n} W_{A,B}$ by Lemma 5.3;

(4) for each pattern $(A,B) \in \mathscr{P}$, there exists a deterministic algorithm outputting an $(\varepsilon/5)$-approximation $\widehat{Z}_{A,B}$ to $Z^{\mathsf{sp}}_{A,B}$ in time $(n/\varepsilon)^{\tau(\Delta)}$ for a factor $\tau = \tau(\Delta)$ by Lemma 5.4.

Combining all above together, we obtain that there exists a deterministic algorithm such that it outputs a quantity $\widehat{Z} := \sum_{(A,B)\in\mathscr{P}} \widehat{Z}_{A,B}$ in time $2^q (n/\varepsilon)^{\tau(\Delta)}$ and $\widehat{Z}$ is an ε-approximation to $Z_{G,q}(\beta)$, thus concluding the theorem. ☐

5.1. **Estimation by patterns.** The proof of Lemma 5.2 is similar to [JKP20, Lemma 29] for proper colorings, and we show how it adapts the anti-ferromagnetic case.

Proof of Lemma 5.2. It is trivial to see $Z^{\le\theta n} \le \sum_{(A,B)\in\mathscr{P}} \sum_{S\subseteq V:|S|\le\theta n} W_{A,B}(S)$. For the other side, consider a configuration in both $\chi_{A,B}(S)$ and $\chi_{C,D}(T)$ for $(A,B,S) \ne (C,D,T)$ with patterns $(A,B),(C,D) \in \mathscr{P}$ and $|S|,|T| \le \theta n$. It must hold that $(A,B) \ne (C,D)$ (otherwise $S = T$). Then by Proposition 3.4, it holds that

$$\begin{aligned}\sum_{(A,B)\in\mathscr{P}} \sum_{S\subseteq V:|S|\le\theta n} W_{A,B}(S) - Z^{\le\theta n} &\le \sum_{(A,B)\ne(C,D)\in\mathscr{P}} \sum_{S,T\subseteq V:|S|,|T|\le\theta n} \sum_{\sigma\in\chi_{A,B}(S)\cap\chi_{C,D}(T)} w(\sigma)\\ &\le 2^{2q}\left(\sum_{\ell\le\theta n}\binom{2n}{\ell}\right)^2 \left(\frac{q-1}{2}\right)^{2n} q^{2\theta n}\\ &\le 2^{2q} \mathrm{e}^{(4H(\theta/2)+2\theta\ln q-2/q)n} (q/2)^{2n}\\ &\le \mathrm{e}^{-n/(100q)} Z^{\le\theta n}\end{aligned}$$

where we use the inequality $\sum_{\ell\le k}\binom{n}{\ell} \le \mathrm{e}^{nH(k/n)}$ for $k \le n/2$ and a crude bound $Z^{\le\theta n} \ge (q/2)^{2n}$ by considering the number of proper colorings. Thus we conclude the lemma. ☐

5.2. **From sparse subsets to low-rank subsets.** The following claim about the size of a sparse set plays a crucial role in the proof of Lemma 5.3.

Claim 5.5 (a general form of [JKP20, Claim 30])**.** *Consider a bipartite (ρ,α)-expander $G = (V = \mathcal{L}\cup\mathcal{R}, E)$ and $\theta \in (0,1)$ such that $\rho \le \theta/2$ and $\rho\alpha + \theta/2 > 1$. For every $S \in \Gamma^{\mathsf{sp}}(G,\theta)$, it holds that $|S| \le 3\theta n$.*

Proof. Assume that $|S| > 3\theta n$. Let $\gamma_1,\dots,\gamma_k$ be its 3-connected components. Since S is sparse, we know that $|\gamma_i| \le \theta n$ for $i = 1,\dots,k$. Then we partition $\{\gamma_1,\dots,\gamma_k\}$ into Γ_1 and Γ_2 such that $\sum_{\gamma\in\Gamma_1}|\gamma| \ge \theta n$ and $\sum_{\gamma\in\Gamma_2}|\gamma| \ge \theta n$. Let $S_1 := \bigcup_{\gamma\in\Gamma_1}\gamma$ and $S_2 := \bigcup_{\gamma\in\Gamma_2}\gamma$. Without loss of generality, assume that $|S_1\cap\mathcal{L}| \ge \theta n/2$. By the assumption that G is a bipartite (ρ,α)-expander, it holds that

$$|S_1 \cup N_G(S_1)| \ge |(S_1\cap\mathcal{L}) \cup N_G(S_1\cap\mathcal{L})| \ge (\theta/2)n + \rho\alpha n > n.$$

A similar argument shows that $|S_2 \cup N_G(S_2)| > n$, leading to a contradiction that $d_G(S_1,S_2) > 3$. ☐

Proof of Lemma 5.3. By Lemma 2.3 and Corollary 2.5, we assume that statements in Lemma 2.3 and Corollary 2.5 hold (this event occurs with high probability).

Observe that for every $|S| \le \theta n$, $S \in \Gamma^{\mathsf{sp}}(G,\theta)$. Then

$$\sum_{(A,B)\in\mathscr{P}} \sum_{S\subseteq V:|S|\le\theta n} W_{A,B}(S) \le \sum_{(A,B)\in\mathscr{P}} Z^{\mathsf{sp}}_{A,B}.$$

For the other side, consider a sparse set S of size $|S| > \theta n$. By Claim 5.5, it holds that $|S| \le 3\theta n$ for $\theta \ge \frac{8\ln\Delta}{\Delta} > \max\{2\rho, 2(1-\rho\alpha)\}$. Then by Corollary 2.5, it holds that $|N_G(S)| \ge$

$\left(\frac{\Delta}{4+(\Delta-4)|S|/n}-1\right)|S|$. Using the crude bounds $|A|,|B|\le q$, $|A|/|B|\in[1/q,q]$ and $|S\cap\mathcal{R}|-|S\cap\mathcal{L}|\le|S|$, by Lemma 3.2,

$$W_{A,B}(S)\le|A|^n|B|^n\cdot q^{|S|}\left(1-\frac{1-\mathrm{e}^{-\beta}}{q}\right)^{|N_G(S)|}.$$

Thus we calculate that

$$\begin{aligned}
&\sum_{(A,B)\in\mathscr{P}}Z^{\mathsf{sp}}_{A,B}-\sum_{(A,B)\in\mathscr{P}}\sum_{S\subseteq V:|S|\le\theta n}W_{A,B}(S)\\
&\le\sum_{(A,B)\in\mathscr{P}}\sum_{S\in\Gamma^{\mathsf{sp}}:|S|\ge\theta n}|A|^n|B|^n\exp\left(|S|\ln q-\frac{1-\mathrm{e}^{-\beta}}{q}|N_G(S)|\right)\\
&\le\sum_{(A,B)\in\mathscr{P}}|A|^n|B|^n\sum_{\ell=\theta n}^{3\theta n}\binom{2n}{\ell}\exp\left(\ell\left(\ln q-\frac{1-\mathrm{e}^{-\beta}}{q}\left(\frac{\Delta}{4+(\Delta-4)\ell/n}-1\right)\right)\right)\\
&\le\mathrm{e}^{-\frac{C(1-\mathrm{e}^{-\beta})}{q}n}\sum_{(A,B)\in\mathscr{P}}|A|^n|B|^n
\end{aligned}$$

for some absolute constant $C>0$, where we use the inequality $\sum_{\ell\le k}\binom{n}{\ell}\le\mathrm{e}^{nH(k/n)}$ for $k\le n/2$ and the fact that there exists some $C'>0$ for every $x\in[\theta,3\theta]$, $x(\Delta/(4+(\Delta-4)x)-1)\ge C\min\{\Delta x,1\}$ for some $C>0$ and the last inequality holds by a crude bound $\sum_{S\subseteq V:|S|\le\theta n}W_{A,B}(S)\ge|A|^n|B|^n$ Thus we conclude the lemma. □

5.3. **Abstract polymer models.** In this part, we provide our algorithm in Lemma 5.4 to approximately compute $Z^{\mathsf{sp}}_{A,B}$ for each pattern $(A,B)\in\mathscr{P}$. The main tool is the *abstract polymer model* introduced in Section 2.4. Fix a pattern (A,B). We construct the abstract polymer model $\mathcal{M}^{\theta}_{A,B}=(\mathcal{C},\sim)$ as following. For every connected subgraph $S\subseteq V$ in G^3 of size $|S|\le\theta n$, it induces a polymer $\gamma=(V_\gamma,w_\gamma)$ with $V_\gamma=S$ and

$$w_\gamma:=\frac{W_{A,B}(V_\gamma)}{|A|^n|B|^n}.$$

For two polymers γ_1,γ_2, we say $\gamma_1\sim\gamma_2$ if the induced subgraph $G^3[\gamma_1\cup\gamma_2]$ is connected.

For the model of proper colorings, [JKP20] shows that the partition function of the polymer model is somehow a 'normalized' number of proper colorings at low ranks. For the anti-ferromagnetic Potts model, we put the following proposition showing the relation between $\Xi(\mathcal{M}^{\theta}_{A,B})$ and $Z^{\mathsf{sp}}_{A,B}$ stated in [GGS21] and for completeness we provide the proof.

Proposition 5.6. *For every pattern $(A,B)\in\mathscr{P}$, it holds that*

$$Z^{\mathsf{sp}}_{A,B}=|A|^n|B|^n\cdot\Xi(\mathcal{M}^{\theta}_{A,B}).$$

Proof. For a subset $S\subseteq V$, observe that (for simplicity, let $\Lambda=S\cup N_G(S)$)

$$\begin{aligned}
\frac{W_{A,B}(S)}{|A|^n|B|^n}&=\frac{1}{|A|^n|B|^n}\sum_{\sigma\in\chi_{A,B}(S)}\mathrm{e}^{-\beta|m_G(\sigma)|}\\
&=\frac{1}{|A|^n|B|^n}\sum_{\sigma_\Lambda\in[q]^\Lambda:\exists\sigma\in\chi_{A,B}(S),\sigma(\Lambda)=\sigma_\Lambda}\mathrm{e}^{-\beta|m_G(\sigma_\Lambda)|}|A|^{n-|\Lambda\cap\mathcal{L}|}|B|^{n-|\Lambda\cap\mathcal{R}|}\\
&=\frac{\sum_{\sigma_\Lambda\in[q]^\Lambda:\exists\sigma\in\chi_{A,B}(S),\sigma(\Lambda)=\sigma_\Lambda}\mathrm{e}^{-\beta|m_G(\sigma_\Lambda)|}}{|A|^{|\Lambda\cap\mathcal{L}|}|B|^{|\Lambda\cap\mathcal{R}|}}.
\end{aligned}$$

Assume that 3-connected components of S are $S_1, \dots, S_k$. Then it holds that

$$\frac{W_{A,B}(S)}{|A|^n|B|^n} = \prod_{i=1}^{k} \frac{W_{A,B}(S_i)}{|A|^n|B|^n}.$$

Note that there is a one-to-one correspondence between sparse subsets and compatible polymers. Thus we conclude the proposition. □

5.3.1. *Verify Kotecký-Preiss condition.* Now we verify the Kotecký-Preiss condition (Condition 2.8) and thus we obtain an **FPTAS** for computing $\Xi(\mathcal{M}^{\theta}_{A,B})$ by Theorem 2.9. The following upper bound for the weight of a polymer is the key ingredient.

Proposition 5.7. *For positive integers* $q, \Delta \geq 4$ *and real numbers* $\beta > 0, \theta \in (0,1)$, *the following holds with high probability over* $G \sim \mathcal{G}^{\mathsf{bip}}_{n,\Delta}$ *in the abstract polymer model* $\mathcal{M}^{\theta}_{A,B} = (\mathcal{C}, \sim)$ *induced by every pattern* $(A,B) \in \mathscr{P}$ *for the anti-ferromagnetic q-Potts model on* G *at inverse temperature* β *together with* θ*:*

$$\forall \gamma \in \mathcal{C}, \quad |w_\gamma| \leq \exp\left(\left(\ln q - \frac{1-\mathrm{e}^{-\beta}}{q}\left(\frac{\Delta}{4+(\Delta-4)\theta} - 1\right)\right)|\gamma|\right).$$

Proof. We assume that the statement in Corollary 2.5 holds (this event occurs with high probability). For a polymer $\gamma = (V_\gamma, w_\gamma)$, by the crude bound $|A|, |B| \leq q$, $|A|/|B| \in [1/q, q]$ and $|V_\gamma \cap \mathcal{R}| - |V_\gamma \cap \mathcal{L}| \leq |\gamma|$, it holds that

$$\begin{aligned}|w_\gamma| &\leq q^{|\gamma|}\left(1 - \frac{1-\mathrm{e}^{-\beta}}{q}\right)^{|N_G(V_\gamma)|} \\ &\leq \exp\left(|\gamma|\ln q - \frac{1-\mathrm{e}^{-\beta}}{q}|N_G(V_\gamma)|\right) \\ &\leq \exp\left(\left(\ln q - \frac{1-\mathrm{e}^{-\beta}}{q}\left(\frac{\Delta}{4+(\Delta-4)\theta} - 1\right)\right)|\gamma|\right)\end{aligned}$$

By Corollary 2.5. Thus we obtain the upper bound. □

Now we are ready to prove Lemma 5.4.

Proof of Lemma 5.4. Fix a pattern $(A,B) \in \mathscr{P}$ and we construct the abstract polymer model $\mathcal{M}^{\theta}_{A,B} = (\mathcal{C}, \sim)$ as before. To apply Theorem 2.9, we pick $g(\gamma) = |\gamma|$ and verify Items 1 to 3 in Theorem 2.9. Obviously Items 1 and 2 hold by definition and our construction. Then we verify the Kotecký-Preiss condition (Condition 2.8) with g.

For a polymer $\gamma \in \mathcal{C}$, we enumerate $\gamma' \sim \gamma$ by the following steps:

(1) firstly we enumerate a vertex v such that $d_G(v, V_\gamma) \leq 3$;
(2) then we enumerate a natural number ℓ denoting the size of γ';
(3) lastly we enumerate all 3-connected subgraphs of size ℓ containing v.

By Proposition 5.7 and the fact that there are at most $(\mathrm{e}\Delta^3)^\ell$ connected subgraphs in G^3 containing v of size ℓ, it holds that

$$\begin{aligned}\sum_{\gamma' \sim \gamma} |w_{\gamma'}| \mathrm{e}^{|\gamma'| + g(\gamma')} &\leq \sum_{v \in V : d_G(v, V_\gamma) \leq 3} \sum_{v \ni \gamma'} |w_{\gamma'}| \mathrm{e}^{|\gamma'| + g(\gamma')} \\ &\leq \Delta^3 |\gamma| \sum_{\ell=1}^{\infty} \exp(\ell + 3\ell \ln \Delta + 2\ell - (5 + 9\ln\Delta)\ell)\end{aligned}$$

$$\leq \Delta^3|\gamma| \sum_{\ell=1}^{\infty} \Delta^{-6\ell}$$
$$\leq g(\gamma).$$

Then by Theorem 2.9, there exists a deterministic algorithm to output an ε-approximation to $\Xi(\mathcal{M}_{A,B})$ running in time $(n/\varepsilon)^{\tau(\Delta)}$ for a computable function $\tau : \mathbb{R} \to (0, +\infty)$ depending only on Δ. Applying this algorithm to estimate $\Xi(\mathcal{M}_{A,B})$ and by Proposition 5.6, we obtain the desired algorithm. □

6. CONCLUSION

In this work, we study the anti-ferromagnetic q-Potts models on random regular bipartite graphs $\mathcal{G}^{\mathsf{bip}}_{n,\Delta}$. Though the celebrated single-site Glauber dynamics exhibits a slow mixing time, there still might be an **FPTAS** for approximating the partition function $Z_{G,q}(\beta)$ in the non-uniqueness regime. Both negative and positive results rely on the geometric properties of the model. We state some discussion and conjectures here.

6.1. Geometric properties of anti-ferromagnetic Potts models.

In Section 3, we demonstrate the configuration space according to ranks when β is far away from the uniqueness regime for sufficiently large q. The key observation is that, for anti-ferromagnetic Potts models on bipartite expanders, when the inverse temperature β is sufficiently large, most of configurations are concentrated around some ground states characterized by different patterns. It has been mentioned above that such properties are inspired by the existence of long-range order of anti-ferromagnetic 3-state Potts models on lattices $\mathbb{Z}^d$ conjectured by [Kot85] and answered by [FS19] for sufficiently large d. However, it still remains an understanding when such phase transition occurs for anti-ferromagnetic q-Potts models on random regular bipartite graphs and we ask the following question.

Question 6.1. *For anti-ferromagnetic q-state Potts models on random Δ-regular bipartite graphs, when does the model undergo a phase transition?*

We remark here that Question 6.1 is important since our results in this work actually rely on the property that the model is somehow in an ordered phase.

6.2. Torpid and rapid mixing of Markov chains.

Theorem 1.1 shows the torpid mixing of Glauber dynamics for anti-ferromagnetic q-Potts models when β is far away from the uniqueness. It has been shown in [GKRS15] that the local-update Markov chain (including the Glauber dynamics) exhibits torpid mixing for 3-colorings on lattices by the existence of phase transition. We conjecture that the coexistence of different phases implies the torpid mixing of this family of Markov chains.

Conjecture 6.2. *For anti-ferromagnetic q-Potts models where the phase transition occurs, the local-update Markov chains (including the single-site Glauber dynamics P^{GD}) exhibit exponential mixing time at sufficiently low temperature.*

On the other side, in spite of the torpid mixing of P^{GD}, it remains open whether there exists a fast Markov chain to design an **FPAUS** for $\mu_{G,q;\beta}$ (for example, polymer dynamics in [GGS21]).

Conjecture 6.3. *For anti-ferromagnetic q-Potts models on random Δ-regular $2n$-vertex bipartite graphs at $\beta > 0$, with high probability there exists an* **FPAUS** *for $\mu_{G,q;\beta}$ running in time* $\mathrm{poly}(n, \Delta)$.

6.3. Approximate counting.

To estimate the partition function $Z_{G,q}(\beta)$ of anti-ferromagnetic q-state Potts model on $\mathcal{G}^{\mathsf{bip}}_{n,\Delta}$, we apply the method of abstract polymer models at low temperature far away from the uniqueness threshold. We put the following conjecture on counting anti-ferromagnetic q-Potts models on random regular bipartite graphs.

Conjecture 6.4. *For anti-ferromagnetic q-Potts models on random Δ-regular bipartite graphs at $\beta > 0$, with high probability there exists an* **FPTAS** *for $Z_{G,q}(\beta)$.*

Meanwhile, the Erdős-Rényi random bipartite graph $\mathcal{G}(n,n,p)$ is another important random models in combinatorics and theoretic computer science. We generate $G = (\mathcal{L}, \mathcal{R}, E) \sim \mathcal{G}(n,n,p)$ of size $|\mathcal{L}| = |\mathcal{R}| = n$ by picking each edge $(u,v) \in \mathcal{L} \times \mathcal{R}$ in E with probability p independently at random. We put the following conjecture on counting anti-ferromagnetic Potts models on sparse Erdős-Rényi random bipartite graphs.

Conjecture 6.5. *For anti-ferromagnetic q-Potts models on Erdős-Rényi random bipartite graph $\mathcal{G}(n,n,d/n)$ at $\beta > 0$, with high probability there exists an* **FPTAS** *for $Z_{G,q}(\beta)$.*

Remark 6.6. The technique we apply for random regular bipartite graphs seems to fail on the Erdős-Rényi random bipartite graphs, since it is seemingly impossible to verify the convergent criterion when the underlying graph is not a bipartite expander. Also, it remains a barrier to design the approximating algorithm on unbounded-degree graphs (Theorem 2.9 has a strict restriction on the assumption that the degree Δ is a constant).

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