Properties of Non-Backtracking Matrix

1 Hashimoto Matrix

Given a graph $G = (V, E)$, let $\vec{E}$ be the oriented edge set generated from E by separating each edge $(u, v) \in E$ into two oriented edges $u \to v$ and $v \to u$. The *Hashimoto matrix* (non-backtracking matrix) $H_G \in \mathbb{R}^{\vec{E}\times\vec{E}}$ of G is defined as

$$H_G(u \to v, x \to y) = \begin{cases} 1 & v = x \wedge u \neq y \\ 0 & \text{otherwise} \end{cases}.$$

We would show some properties of non-backtracking matrices.

2 Ihara-Zeta Function and Determinant of H_G

For a graph $G = (V, E)$, *a reduced closed walk of* G is a path $p = \{v_0, \ldots, v_{\ell-1}\}$ satisfying for all $0 \le i \le \ell - 1$,

$$(v_i, v_{(i+1) \bmod \ell}) \in E$$
$$v_i \neq v_{(i+2) \bmod \ell}$$

Note that, any cyclic rotation of p is also a reduced closed walk. Let $\langle p \rangle$ be the cyclic group induced by p. We consider every two elements in the same cyclic group are equal, and so we pick any of $\langle p \rangle$ as a prime reduced closed walk. Obviously there is no reduced closed walk of length < 3.

We define *Ihara-Zeta function* $\zeta_G(u)$ as

$$\zeta_G(u) = \prod_p \frac{1}{1 - u^{L(p)}}$$

where p ranges over all prime reduced closed walks and $L(p)$ is the length of p. Let $\Gamma_G(\ell)$ be the collection of all prime reduced closed walks with length ℓ. Thus we know

$$\zeta_G(u) = \prod_{\ell \ge 1} \left(\frac{1}{1 - u^\ell} \right)^{|\Gamma_G(\ell)|} = \exp\left(\sum_{\ell=1}^{\infty} |\Gamma_G(\ell)| \frac{u^\ell}{\ell} \right).$$

On the other hand, it holds that

$$\mathrm{Tr}\left(H_G^\ell\right) = |\Gamma_G(\ell)|, \quad \forall \ell \in \mathbb{N}.$$

Then, by Jacobi's identity, we know

$$\zeta_G(u) = \exp\left(\sum_{\ell=1}^{\infty} |\Gamma_G(\ell)| \frac{u^\ell}{\ell} \right) = \exp\left(\sum_{\ell=1}^{\infty} \mathrm{Tr}\left(H^\ell\right) \frac{u^\ell}{\ell} \right) = \frac{1}{\det(I - uH_G)}.$$

Let A_G be the adjacent matrix of G and $D_G = \mathrm{diag}\left(\deg_G\right)$ be the degree matrix of G. The following identity holds for Ihara-Zeta function.

Lemma 2.1 (Ihara-Bass's formula)**.** *Given a graph $G = (V, E)$ and $H_G, A_G, D_G, \zeta_G(u)$ defined as above, it holds that*

$$\zeta_G(u) = \frac{1}{\det(I - uH_G)} = \frac{1}{(1 - u^2)^{|E|-|V|} \det(I - uA_G + u^2(D_G - I))}.$$

3 Hashimoto Matrix of Erdős-Rényi Graphs

Now we consider the Hashimoto matrix of a random graph $G \sim \mathcal{G}(n, \alpha/n)$.

Lemma 3.1 (Theorem 3 in [BLM18]). *Given a fixed parameter $\alpha > 0$, let $G = (V, E)$ be a random graph generated from Erdős-Rényi graph $\mathcal{G}(n, \alpha/n)$. Then with probability tending to 1 as $n \to \infty$, the eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ of its Hashimoto matrix H_G satisfy:*

$$\lambda_1 = \alpha + o(1), \quad |\lambda_2| \leq \sqrt{\alpha} + o(1).$$

References

[BLM18] Charles Bordenave, Marc Lelarge, and Laurent Massoulié. Nonbacktracking Spectrum of Random Graphs: Community Detection and Nonregular Ramanujan Graphs. *The Annals of Probability*, 46(1):1–71, 2018. 2