Christoffel-Darboux Identity

Zhidan Li

1 The Independence Polynomial

Given a graph $G = (V(G), E(G))$, we say a subset $S \subseteq V$ of vertices is an *independent set* if for all two distinct $u, v \in S$, u is not adjacent to v, *i.e.*, $(u, v) \notin E$. Let $\mathcal{I}(G)$ be the collection of all independent sets on G. The *independence polynomial* of G is defined by

$$I(G, x) = \sum_{S \in \mathcal{I}(G)} x^{|S|}. \tag{1}$$

For every subset of vertices $S \subseteq V$, we use $[S] := \{u \in V : d_G(u, S) \leq 1\}$ to denote the union set of S and the neighborhood of S. It is obvious to see the following identities hold:

- For two disjoint graphs G_1 and G_2,

$$I(G_1 \cup G_2, x) = I(G_1, x) \cdot I(G_2, x).$$

- For every vertex $v \in V$,

$$I(G, x) = I(G \setminus \{v\}, x) + xI(G \setminus [v], x).$$

2 Christoffel-Darboux Identity for the Independence Polynomial

Now we show the so-called *Christoffel-Darboux identity* for the independence polynomial on every graph.

Theorem 2.1 (Theorem 1.3 in [Ben18]). *Let $G = (V(G), E(G))$ be a graph and $u, v \in V(G)$ be two vertices. Let $\mathcal{B}_{u,v}$ be the set of induced connected, bipartite graphs containing the vertices u and v. Then*

$$I(G \setminus \{u\}, x)I(G \setminus \{v\}, x) - I(G, x)I(G \setminus \{u, v\}, x) = \sum_{H \in \mathcal{B}_{u,v}} (-1)^{d_H(u,v)+1} x^{|V(H)|} I(G \setminus [H], x)^2.$$

Proof. Let

$$\begin{aligned} \mathcal{I}_1 &:= \mathcal{I}(G \setminus \{u\}) \times \mathcal{I}(G \setminus \{v\}), \\ \mathcal{I}_2 &:= \mathcal{I}(G) \times \mathcal{I}(G \setminus \{u, v\}). \end{aligned}$$

Then we see the LHS of the identity is equal to

$$\sum_{(A,B) \in \mathcal{I}_1} x^{|A|+|B|} - \sum_{(A,B) \in \mathcal{I}_2} x^{|A|+|B|}.$$

Note that, for every $(A, B) \in \mathcal{I}_1$ and $u, v \notin A \cup B$, it holds that $(A, B) \in \mathcal{I}_2$ since $A \cup B \in \mathcal{I}(G \setminus \{u, v\}) \subseteq \mathcal{I}(G)$. Conversely, for every $(C, D) \in \mathcal{I}_2$ with $u, v \notin C \cup D$, it also holds that $(C, D) \in \mathcal{I}_1$. And then they will cancel each other.

When $u \in B$ but $v \notin A$, it holds that $(B, A) \in \mathcal{I}_2$. When $v \in A$ but $u \notin B$, it holds that $(A, B) \in \mathcal{I}_2$. Then we consider the following two sets:

$$\mathcal{I}_1' := \{(A, B) \in \mathcal{I}_1 \ : \ u, v \in A \cup B\},$$
$$\mathcal{I}_2' := \{(A, B) \in \mathcal{I}_2 \ : \ u, v \in A \cup B\}.$$

Then the LHS of the identity is equal to

$$\sum_{(A,B)\in\mathcal{I}_1'} x^{|A|+|B|} - \sum_{(A,B)\in\mathcal{I}_2'} x^{|A|+|B|}.$$

Note that, if $z \in A \cap B$, we know $G[A \triangle B]$ is a bipartite graph, and the two parts can be $A \setminus B$ and $B \setminus A$. And we know u, v are always part of a bipartite graph.

Suppose that $(A, B) \in \mathcal{I}_1'$, and u and v are not in the same connected component. By switching the colors of the component containing v, we see the new pair $(A', B') \in \mathcal{I}_2'$. Then we see every pair is cancelled. Now we suppose that they are in the same component of $G[A \cup B]$. Let

$$\mathcal{I}_1'' := \left\{(A, B) \in \mathcal{I}_1' \ : \ u, v \text{ is connected in } G[A \cup B]\right\},$$
$$\mathcal{I}_2'' := \left\{(A, B) \in \mathcal{I}_2' \ : \ u, v \text{ is connected in } G[A \cup B]\right\}.$$

Observe that, if A, B are independent sets of G, $u, v \in A \cup B$ and u, v are connected in $G[A \cup B]$, $(A, B) \in \mathcal{I}_1''$ or $(A, B) \in \mathcal{I}_2''$. When $(A, B) \in \mathcal{I}_1''$, $d_{G[A\cup B]}(u, v)$ is odd and otherwise is even. Let $P(A, B)$ be such the connected component containing u, v. Then

$$\begin{aligned}\sum_{(A,B)\in\mathcal{I}_1''\cup\mathcal{I}_2''} (-1)^{d_P(u,v)+1} x^{|P(A,B)|} x^{|A|+|B|-|P(A,B)|} &= \sum_{H\in\mathcal{B}_{u,v}} (-1)^{d_H(u,v)+1} x^{|V(H)|} \left(\sum_{A,B\in\mathcal{I}(G\setminus[H])} x^{|A|} x^{|B|} \right) \\ &= \sum_{H\in\mathcal{B}_{u,v}} (-1)^{d_H(u,v)+1} x^{|V(H)|} I(G \setminus [H], x)^2.\end{aligned}$$

□

As a special case, when G is a tree, we cover the result discovered by Gutman [Gut91].

Corollary 2.2 (Theorem 1 in [Gut91])**.** *Let T be a tree and u, v be two vertices. Let P be the only path connecting u and v. Then we know*

$$I(T \setminus \{u\}, x) I(T \setminus \{v\}, x) - I(T, x) I(T \setminus \{u, v\}, x) = -(-x)^{d_T(u,v)} I(T \setminus P, x) I(T \setminus [P], x).$$

References

[Ben18] Ferenc Bencs. Christoffel–Darboux Type Identities for the Independence Polynomial. *Combinatorics, Probability and Computing*, 27(5):716–724, 2018. 1

[Gut91] Ivan Gutman. An Identity for the Independence Polynomials of Trees. *Publ. Inst. Math.(Belgrade)*, 50:19–23, 1991. 2