Canonical Paths, Multi-Commodity Flows and Windability

1 Canonical Paths and Multi-Commodity Flow

Fix a distribution μ over the state space Ω. Let P be a Markov transition kernel which is reversible with respect to μ. Define the mixing time t_{mix} as

$$t_{\mathrm{mix}}(P, x, \varepsilon) := \inf \left\{ t \geq 0 : \ \mathcal{D}_{\mathrm{TV}} \left(P^t(x, \cdot) \parallel \mu \right) \leq \varepsilon \right\}$$

where $\mathcal{D}_{\mathrm{TV}}(\cdot \parallel \cdot)$ is the total variation distance between two distributions. Assume that the eigenvalues of P are $1 = \lambda_1 > \lambda_2 \geq \ldots \geq \lambda_n \geq -1$. Let $\lambda' = \max\{\lambda_2, |\lambda_n|\}$. The following proposition upper bounds the mixing time of P.

Proposition 1.1 (Proposition 1 in [Sin92]). *The following inequalities hold:*

1. $t_{\mathrm{mix}}(P, x, \varepsilon) \leq \frac{1}{1-\lambda'} \left(\log \frac{1}{\mu(x)} + \log \frac{1}{\varepsilon} \right)$.
2. $\max_{x \in \Omega} t_{\mathrm{mix}}(\varepsilon) \geq \frac{\lambda'}{2(1-\lambda')} \log \frac{1}{2\varepsilon}$.

To bound λ', we introduce the method of *canonical paths and multi-commodity flows*. Let $\mathcal{G} = (\mathcal{V} = \Omega, \mathcal{E})$ be the transition graph of P. *Canonical paths* Γ *from* $\Omega_x \subseteq \Omega$ *to* $\Omega_y \subseteq \Omega$ is a family of simple paths on $\mathcal{G}$ equipped with weights $w : \Gamma \to \mathbb{R}_{\geq 0}$ satisfying

$$\sum_{\gamma \in \Gamma : \gamma \text{ from } x \text{ to } y} w(\gamma) = \mu(x)\mu(y), \quad \forall x \in \Omega_x, y \in \Omega_y.$$

Define the *congestion* $\rho(\Gamma)$ *of* Γ as

$$\rho(\Gamma) := \max_{\sigma, \tau \in \Omega : (\sigma, \tau) \in \mathcal{E}} \frac{1}{\mu(\sigma) P(\sigma, \tau)} \sum_{\gamma \in \Gamma : \gamma \ni (\sigma, \tau)} w(\gamma).$$

The following lemma connects the mixing time with the congestion.

Lemma 1.2 ([Sin92]). *For every canonical paths* Γ *from* Ω *to* Ω*, every* $\sigma \in \Omega$ *and non-negative integer* $t \in \mathbb{N}$*, it holds that*

$$\mathcal{D}_{\mathrm{TV}} \left(P^t(\sigma, \cdot) \parallel \mu \right) \leq \frac{1}{2\sqrt{\mu(\sigma)}} \exp\left(-\frac{t}{n\rho(\Gamma)} \right).$$

On the other hand, the phenomenon of rapid mixing also implies low congestion.

Lemma 1.3 (Theorem 8 in [Sin92]). *Let* $\tau = \max_{\sigma \in \Omega} t_{\mathrm{mix}}(P, \sigma, 1/4)$ *and* ρ *be the minimal congestion over all canonical paths from* Ω *to* Ω*. Then it holds that*

$$\rho \leq 16 n \tau.$$

2 Holant Problems and Windability

Now let $G = (V, E)$ be a graph. Let $\mathcal{E}$ be the collection of half-edges on G, *i.e.*,

$$\mathcal{E} := \{(e_u, e_v) \mid e = (u, v) \in E\}.$$

For every vertex $v \in V$, let $\mathcal{E}(v)$ be the half-edges incident to v.

An instance of a Holant problem is a tuple $\Lambda = (G = (V, E), \{f_v\}_{v \in V})$ where for every $v \in V$, $f_v : \{0,1\}^{\mathcal{E}(v)} \to \mathbb{R}_+$ is a function. For every configuration $\sigma \in \{0,1\}^{\mathcal{E}}$, we define the weight of σ as

$$w_\Lambda(\sigma) := \prod_{v \in V} f_v(\sigma|_{\mathcal{E}(v)}).$$

For a configuration $\sigma \in \{0,1\}^{\mathcal{E}}$, let $d(\sigma)$ be the number of edges $e = (u, v)$ such that $\sigma(e_u)$ disagrees with $\sigma(e_v)$, *i.e.*,

$$d(\sigma) := |\{e = (u, v) \in E \mid \sigma(e_u) \neq \sigma(e_v)\}|.$$

For every $k \geq 0$, let $\Omega_k := \left\{\sigma \in \{0,1\}^{\mathcal{E}} \,\middle|\, d(\sigma) = k\right\}$ and $Z_k(\Lambda) := \sum_{\sigma \in \Omega_k} w_\Lambda(\sigma)$.

2.1 Symmetric and Windable functions

Given an indexing set J, for every $x \in \{0,1\}^J$, define $|x|$ as the Hamming weight of x, *i.e.*, $|x| = \sum_{i \in J} x_i$. A function $f : \{0,1\}^J \to \mathbb{R}_+$ is *symmetric* if the value of the function only depends on the Hamming weight of its input. Thus, for a symmetric function $f : \{0,1\}^J \to \mathbb{R}_+$ with $|J| = d$, we write it as $f = [f_0, f_1, \ldots, f_d]$, where f_i is the value of f on inputs with Hamming weight i.

For a function $f : \{0,1\}^J$ and a partial assignment $\tau \in \{0,1\}^I$ where $I \subseteq J$, we define the pinning of f by τ as the function $G : \{0,1\}^{J \setminus I} \to \mathbb{R}_+$ such that for every $\sigma \in \{0,1\}^{J \setminus I}$, $G(\sigma) = F(\sigma \cup \tau)$. For a function $f : \{0,1\}^J \to \mathbb{R}_+$, we define its *complement function* $\overline{f}$ as $\overline{f}(x) := f(J \setminus x)$. Note that for a symmetric function $f = [f_0, \ldots, f_d]$, its complement function $\overline{f}$ is $\overline{f} = [f_d, f_{d-1}, \ldots, f_0]$.

In [McQ13], a special family of symmetric functions called *windable functions* are introduced.

Definition 2.1 (Windable Functions)**.** For any finite indexing set J and any configuration $x \in \{0,1\}^J$, define $\mathcal{M}_x$ as the set of partitions of $\{i \mid x_i = 1\}$ into pairs and at most one singleton. A function $F : \{0,1\}^J \to \mathbb{R}_+$ is *windable* if there exist values $B(x, y, M) \geq 0$ for all $x, y \in \{0,1\}^J$ and all $M \in \mathcal{M}_{x \oplus y}$ satisfying:

1. $F(x)F(y) = \sum_{M \in \mathcal{M}_{x \oplus y}} B(x, y, M)$ for all $x, y \in \{0,1\}^J$.

2. $B(x, y, M) = B(x \oplus S, y \oplus S, M)$ for all $x, y \in \{0,1\}^J$ and all $S \in M \in \mathcal{M}_{x \oplus y}$.

The following result in [McQ13, HLZ16] shows the Holant problems equipped with windable functions can be efficiently computed.

Theorem 2.2 (Theorem 3 in [HLZ16])**.** *There exists an* **FPRAS** *to compute the partition function $Z(\Lambda)$ for instances $\Lambda = (G = (V, E), \{f_v\}_{v \in V})$ with $|V| = n$, if it holds that:*

1. *The instance is self-reducible in the sense of [JVV86].*

2. *For every $v \in V$, the function f_v is windable.*

3. *$Z_2(\Lambda)/Z_0(\Lambda) = n^{O(1)}$.*

The **FPRAS** in Theorem 2.2 is a metropolis Markov chain over state $\Omega_0 \cup \Omega_2$. For every two configurations $\sigma, \tau \in \Omega$, the transition probability $P'(\sigma,\tau)$ is defined as

$$P'(\sigma,\tau) = \begin{cases} \frac{2}{n^2}\min\left\{1, \frac{w_\Lambda(\tau)}{w_\Lambda(\sigma)}\right\} & |\sigma \oplus \pi| = 2 \\ 1 - \frac{2}{n^2}\sum_{\rho:|\sigma\oplus\rho|=2}\min\left\{1, \frac{w_\Lambda(\rho)}{w_\Lambda(\sigma)}\right\} & \sigma = \tau \\ 0 & \text{otherwise} \end{cases}$$

and $P = \frac{1}{2}(I + P')$. To prove Theorem 2.2 we apply the canonical paths and for completeness we include it in Appendix A.

2.1.1 Windability for symmetric functions

Usually it is hard to verify the windability by definition. For symmetric functions, we have another way to verify it.

Definition 2.3. A function $H : \{0,1\}^J \to \mathbb{R}_+$ has a *2-decomposition* if there are values $D(x, M) \geq 0$ where x ranges over $\{0,1\}^J$ and M ranges over partitions of J into pairs and at most one singleton such that

1. $H(x) = \sum_M D(x, M)$ for all x where the sum ranges over all partitions of J into pairs and at most one singleton.
2. $D(x, M) = D(x \oplus S, M)$ for all x, M and all $S \in M$.

Lemma 2.4 (Lemma 5 in [HLZ16])**.** *A function F is windable, if and only if for all pinnings G of F, $G \cdot \overline{G}$ has a 2-decomposition.*

References

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A Construction and Analysis of Canonical Paths

Now we prove Theorem 2.2. Given an instance $\Lambda = (G = (V, E), \{f_v\}_{v\in V}$ where $|V| = n$ and f_v is windable for all $v \in V$, consider the distribution $\mu = \mu_\Lambda$ over $\Omega = \Omega_0 \cup \Omega_2$ defined as

$$\mu_\Lambda(\sigma) = \frac{w_\Lambda(\sigma)}{Z_0 + Z_2}, \forall \sigma \in \Omega.$$

As described above, our chain is define as

$$P(\sigma,\tau) = \begin{cases} \frac{1}{n^2}\min\left\{1, \frac{w_\Lambda(\tau)}{w_\Lambda(\sigma)}\right\} & |\sigma \oplus \pi| = 2 \\ 1 - \frac{1}{n^2}\sum_{\rho:|\sigma\oplus\rho|=2}\min\left\{1, \frac{w_\Lambda(\rho)}{w_\Lambda(\sigma)}\right\} & \sigma = \tau \\ 0 & \text{otherwise} \end{cases}$$

and we use $\mathcal{G}(\Omega, \mathcal{E})$ to denote the transition graph of P. Now what we need to do is to construct canonical paths Γ with $\rho(\Gamma) \le \frac{n^3}{\rho_\Lambda(\Omega_0)^2}$.

A.1 Construction of canonical paths

Now we construct the canonical paths as the following steps. Firstly we construct the paths with weighted flow from Ω_0 to Ω and then based on them we construct the canonical paths from Ω to Ω.

A.1.1 Paths from Ω_0 to Ω

Let $\sigma \in \Omega_0$ and $\tau \in \Omega$ be two configurations. Furthermore let $z = \sigma \oplus \tau$. Consider a tuple

$$\left(M_v \in \mathcal{M}_{z|_{\mathcal{E}(v)}}\right)_{v \in V}$$

and we define T as the set of singletons in $\bigcup_{v \in V} M_v$, *i.e.*,

$$T := \{S \in M_v \mid v \in V, S \text{ is a singleton}\}.$$

It is not hard to see $|T|$ is even. Then we partition T into pairs. Denote this partition by M'. Define $M := \bigcup_{v \in V} M_v \cup M' \in M_z$. We say M is the partition induced by $\left(M_v \in \mathcal{M}_{z|_{\mathcal{E}(v)}}\right)_{v \in V}$.

Under the terms described as above, we construct a canonical path $\gamma_{\sigma,\tau,M}$ as follows. Firstly we construct a graph $G_{M,z} = (V_z, E_M)$ with

$$V_z = \{e_v \in \mathcal{E} \mid z(e_v) = 1\},$$
$$E_M = M \cup \{(e_u, e_v) \in V_z \times V_z \mid (u, v) \in E\}.$$

Observe that $G_{M,z}$ is a union of disjoint cycles and a path. We recursively choose an order of edges $\{e_1, \dots, e_m\}$ in E_M as follows:

- If there is an unique path $P = (e_1, \dots, e_k)$, then start from e_1 and choose edges along the path in the same order. After this, we remove the path P.
- If there is no path, then choose a cycle $C = \{e_1, e_2, \dots, e_k, e_1\}$ where $(e_1, e_2) \in M$. Then start at e_1 and choose edges along the cycle. After this, remove C.

This order induces an order in M. We denote this order by $S_1, \dots, S_t$ where $S_k \in M$ is a pair of half-edges.

For every $k = 0, 1, \dots, t$, let $E_k = \bigcup_{i=1}^{k} S_k$. We construct $\gamma_{\sigma,\tau,M}$ as

$$\sigma = \sigma \oplus E_0 \to \sigma \oplus E_1 \to \cdots \to \sigma \oplus E_t = \tau$$

and equip the path with weight

$$w(\gamma_{\sigma,\tau,M}) = \prod_{v \in V} B_v\left(\sigma|_{\mathcal{E}(v)}, \tau|_{\mathcal{E}(v)}, M_v\right) / (Z_0 + Z_1)^2$$

where for every $v \in V$, B_v is the set of values in the definition of the windability of f_v.

Then for every $\sigma \in \Omega_0$ and $\tau \in \Omega$, it holds that

$$\begin{aligned}
\sum_{M \in \mathcal{M}_z} w(\gamma_{\sigma,\tau,M}) &= \frac{1}{(Z_0 + Z_2)^2} \sum_{\left(M_v \in \mathcal{M}_{z \cap \mathcal{E}(v)}\right)_{v \in V}} \prod_{v \in V} B_v\left(\sigma|_{\mathcal{E}(v)}, \tau|_{\mathcal{E}(v)}, M_v\right) \\
&= \frac{1}{(Z_0 + Z_2)^2} \prod_{v \in V} \sum_{M_v \in \mathcal{M}_{z \cap \mathcal{E}(v)}} B_v\left(\sigma|_{\mathcal{E}(v)}, \tau|_{\mathcal{E}(v)}, M_v\right) \\
&= \frac{1}{(Z_0 + Z_2)^2} \prod_{v \in V} f_v(\sigma|_{\mathcal{E}(v)}) f_v(\tau|_{\mathcal{E}(v)}) \\
&= \mu_\Lambda(\sigma)\mu_\Lambda(\tau)
\end{aligned}$$

where the last but second equality holds from the definition of windability. We denote the canonical paths constructed as above by Γ_0.

A.1.2 Paths from Ω to Ω

For every $\sigma, \tau \in \Omega$, every $\rho \in \Omega_0$, every $M_1 \in \mathcal{M}_{\sigma\oplus\rho}$ and every $M_2 \in \mathcal{M}_{\rho\oplus\tau}$, we construct a path $\gamma_{\sigma,\tau,\rho,M_1,M_2}$ by concatenating the two paths γ_{σ,ρ,M_1} and γ_{ρ,τ,M_2} (note that it is safe to reverse paths in Γ_0 since the transition graph is undirected). We set the weight as

$$w(\gamma_{\sigma,\tau,\rho,M_1,M_2}) = \frac{w(\gamma_{\sigma,\rho,M1})w(\gamma_{\rho,\tau,M_2})}{\mu_\Lambda(\rho)\mu_\Lambda(\Omega_0)}.$$

We verify that

$$\begin{aligned}\sum_{\rho\in\Omega_0}\sum_{M_1\in\mathcal{M}_{\sigma\oplus\rho}}\sum_{M_2\in\mathcal{M}_{\rho\oplus\tau}} w(\gamma_{\sigma,\tau,\rho,M_1,M_2}) &= \sum_{\rho\in\Omega_0}\sum_{M_1\in\mathcal{M}_{\sigma\oplus\rho}}\sum_{M_2\in\mathcal{M}_{\rho\oplus\tau}} \frac{w(\gamma_{\sigma,\rho,M1})w(\gamma_{\rho,\tau,M_2})}{\mu_\Lambda(\rho)\mu_\Lambda(\Omega_0)} \\ &= \sum_{\rho\in\Omega_0} \frac{\mu_\Lambda(\sigma)\mu_\Lambda(\rho)\mu_\Lambda(\tau)}{\mu_\Lambda(\Omega_0)} \\ &= \mu_\Lambda(\sigma)\mu_\Lambda(\tau).\end{aligned}$$

A.2 Analysis of the congestion

Now we analyze the congestion of Γ.

Lemma A.1 (Lemma 31 in [HLZ16]). *Let $\Gamma = (G = (V, E), \{f_v\}_{v\in V})$ be an instance where every f_v is windable. Then $Z_0Z_4 \le Z_2Z_2$.*

Lemma A.2 (Lemma 32 in [HLZ16]). *Let Γ_0 be the canonical paths from Ω_0 to Ω constructed as above. Then*

$$\rho(\Gamma_0) \le \frac{n^3}{\mu_\Lambda(\Omega_0)}.$$

Proof. For every $X, Y \in \Omega$ with $P(X, Y) > 0$, it holds that

$$\mu_\Lambda(X)P(X, Y) = \frac{1}{n^2}\min\{\mu_\Lambda(X), \mu_\Lambda(Y)\}.$$

Then,

$$\begin{aligned}\frac{1}{\mu_\Lambda(X)P(X, Y)}\sum_{\gamma\in\Gamma_0:\gamma\ni(X,Y)} w(\gamma) &= \frac{n^2}{\min\{\mu_\Lambda(X), \mu_\Lambda(Y)\}}\sum_{\gamma\in\Gamma_0:\gamma\ni(X,Y)} w(\gamma) \\ &\le \frac{n^2}{\mu_\Lambda(Y)}\sum_{\sigma\in\Omega_0,\tau\in\Omega}\ \sum_{\left(M_v\in\mathcal{M}_{z|_{\mathcal{E}(v)}}\right)_{v\in V}:Y\in\gamma_{\sigma,\tau,M}} w(\gamma_{\sigma,\tau,M}) \\ &= \frac{n^2}{w_\Lambda(Y)(Z_0 + Z_2)}\sum_{\sigma\in\Omega_0,\tau\in\Omega}\ \sum_{\left(M_v\in\mathcal{M}_{z|_{\mathcal{E}(v)}}\right)_{v\in V}:Y\in\gamma_{\sigma,\tau,M}}\ \prod_{v\in V} B_v(\sigma|_{\mathcal{E}(v)}, \tau|_{\mathcal{E}(v)}, M_v) \\ &= \frac{n^2}{w_\Lambda(Y)(Z_0 + Z_2)}\sum_{\sigma\in\Omega_0,\tau\in\Omega}\ \sum_{\left(M_v\in\mathcal{M}_{z|_{\mathcal{E}(v)}}\right)_{v\in V}:Y\in\gamma_{\sigma,\tau,M}}\ \prod_{v\in V} B_v(Y|_{\mathcal{E}(v)}, (Y\oplus\sigma\oplus\tau)|_{\mathcal{E}(v)}, M_v) \\ &\le \frac{n^2}{w_\Lambda(Y)(Z_0 + Z_2)}\sum_{\omega\in\Omega}\prod_{v\in V} f_v(Y|_{\mathcal{E}(v)})f_v((Y\oplus\omega)|_{\mathcal{E}(v)}) \\ &\le n^2\frac{Z_0 + Z_2 + Z_4}{Z_0 + Z_2} \\ &\le \frac{n^3}{\mu_\Lambda(\Omega_0)}\end{aligned}$$

where the last inequality holds from Lemma A.1. □

Lemma A.3. *Let* Γ *be the canonical paths from* Ω *to* Ω *constructed as above. Then*

$$\rho(\Gamma) \leq \frac{n^3}{\mu_\Lambda(\Omega_0)^2}.$$

Proof. By definition, we know

$$\begin{aligned}
\rho(\Gamma) &= \max_{(X,Y)} \frac{1}{\mu_\Lambda(X)P(X,Y)} \sum_{\sigma,\tau\in\Omega,\rho\in\Omega_0} \sum_{M_1\in\mathcal{M}_{\sigma\oplus\rho}} \sum_{M_2\in\mathcal{M}_{\rho\oplus\tau}} \mathbb{1}\left[(X,Y)\in(\gamma_{\sigma,\rho,M_1}\cup\gamma_{\rho,\tau,M_2})\right] \frac{w(\gamma_{\sigma,\rho,M1})w(\gamma_{\rho,\tau,M_2})}{\mu_\Lambda(\rho)\mu_\Lambda(\Omega_0)} \\
&= \max_{(X,Y)} \frac{1}{\mu_\Lambda(X)P(X,Y)} \sum_{\sigma,\tau\in\Omega,\rho\in\Omega_0} \sum_{M_1\in\mathcal{M}_{\sigma\oplus\rho}:(X,Y)\in\gamma_{\sigma,\tau,M_1}} \sum_{M_2\in\mathcal{M}_{\rho\oplus\tau}} \frac{w(\gamma_{\sigma,\rho,M1})w(\gamma_{\rho,\tau,M_2})}{\mu_\Lambda(\rho)\mu_\Lambda(\Omega_0)} \\
&= \max_{(X,Y)} \frac{1}{\mu_\Lambda(X)P(X,Y)} \sum_{\sigma,\tau\in\Omega,\rho\in\Omega_0} \sum_{M_1\in\mathcal{M}_{\sigma\oplus\rho}:(X,Y)\in\gamma_{\sigma,\tau,M_1}} \frac{w(\gamma_{\sigma,\rho,M1})\mu_\Lambda(\tau)}{\mu_\Lambda(\Omega_0)} \\
&= \max_{(X,Y)} \frac{1}{\mu_\Lambda(X)P(X,Y)} \sum_{\sigma\rho\in\Omega_0} \sum_{M_1\in\mathcal{M}_{\sigma\oplus\rho}:(X,Y)\in\gamma_{\sigma,\tau,M_1}} \frac{w(\gamma_{\sigma,\rho,M1})}{\mu_\Lambda(\Omega_0)} \\
&= \frac{\rho(\Gamma_0)}{\mu_\Lambda(\Omega_0)} \\
&\leq \frac{n^3}{\mu_\Lambda(\Omega_0)^2}.
\end{aligned}$$

□