APPROXIMATELY COUNTING SIX-VERTEX MODELS

1. Background on Six-vertex Model

The *six-vertex model* is originally built on *Eulerian orientations* of a 4-regular planar graph. In the six-vertex model, we only allow that for every vertex v, the orientations of edges around v satisfy that exactly two edges point inwards and the remaining two edges point outwards. For every vertex, the valid configurations in the six-vertex model around a vertex should be one of the following six cases:

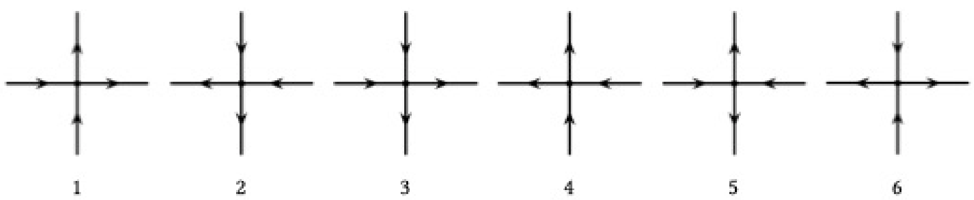

Figure 1. Valid configurations

For a general model, we associate configurations with weights $w_1, w_2, \ldots, w_6$ respectively. We assume the *arrow reversal symmetry* to correspond the physics law, *i.e.*, $w_1 = w_2 = a$, $w_3 = w_4 = b$ and $w_5 = w_6 = c$. We assume that $a, b, c \geq 0$ as in the real physics world. For a 4-regular graph G with edges incident to each vertex labelled from 1 to 4, we define the *partition function* of the six-vertex model as

$$Z(G; a, b, c) := \sum_{\tau \in \Omega^{\mathsf{EO}}(G)} a^{n_1+n_2} b^{n_3+n_4} c^{n_5+n_6}$$

where $\Omega^{\mathsf{EO}}(G)$ is the collection of all Eulerian orientations of G and $n_i = n_i(\tau)$ is the number of vertices of type i under the orientation τ for $i = 1, 2, \ldots, 6$.

1.1. **Six-vertex model as Holant problem.** An alternative view for the six-vertex model is to see it as a type of Holant problem. For a 4-regular graph $G = (V, E)$, consider its edge-vertex incident graph $G' = (U_V, U_E, E')$. To model the orientation of an edge, we introduce the Disequality signature (denoted by $\neq_2$), which receives two boolean bits as input and output whether they are not equal (that is to say, output 1 when input is 01 or 10 and 0 otherwise). We say an orientation on edge $e = \{w, v\}$ is going out w and into v in G if the edge $(u_w, u_e) \in E'$ takes value 1 and $(u_v, u_e) \in E'$ takes value 0. To model the weights on valid configurations, we use a 4-arity signature f, which is of the following matrix form on input $x_1, x_2, x_3, x_4 \in \{0, 1\}$ that

$$M(f) = M_{x_1,x_2,x_4,x_3}(f) = \begin{bmatrix} 0 & 0 & 0 & a \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ a & 0 & 0 & 0 \end{bmatrix}.$$

If we order the left, down, right and up edges incident to a vertex by 1, 2, 3, 4, then we know that the partition function $Z(G; a, b, c)$ is equal to Holant$(G', \neq_2 | f)$. We certify the following families of signatures:

$$\begin{aligned}
\mathcal{F}_{\leq^2} &:= \left\{ f \mid a^2 \leq b^2 + c^2, b^2 \leq a^2 + c^2, c^2 \leq a^2 + b^2 \right\}, \\
\mathcal{F}_{\leq} &:= \{ f \mid a \leq b + c, b \leq a + c, c \leq a + b \}, \\
\mathcal{F}_{=} &:= \{ f \mid c = a + b \} \\
\mathcal{F}_{>} &:= \{ f \mid a, b, c > 0, a > b + c \vee b > a + c \vee c > a + b \}.
\end{aligned}$$

1.2. **Eulerian orientations and Eulerian pairings.** From the graph theoretic term, there is a different view of Eulerian orientations. An *Eulerian partition* of a graph G is a partition of the edges of G into edge-disjoint *circuits*. A *directed Eulerian partition* is an Eulerian partition where every edge-disjoint circuit takes one of the two cyclic orientations. Let $G = (V, E)$ be a 4-regular graph and v be a vertex of G. Let e_1, e_2, e_3, e_4 be the four edges incident to v. A *pairing* ϱ at v is a partition of $\{e_1, e_2, e_3, e_4\}$ into pairs. We use $\cdot \leftrightsquigarrow \cdot$ to denote a pair. There are exactly 3 distinct pairings: $(e_1 \leftrightsquigarrow e_2, e_3 \leftrightsquigarrow e_4)$, $(e_1 \leftrightsquigarrow e_4, e_2 \leftrightsquigarrow e_3)$, $(e_1 \leftrightsquigarrow e_3, e_2 \leftrightsquigarrow e_4)$. We label these cases by symbols ρ_1, ρ_2 and ρ_3. Using this kind of language, an Eulerian partition can be uniquely determined by a family of pairings $\varphi = \{\varrho_v\}_{v\in V}$ where $\varrho_v \in \{\rho_1, \rho_2, \rho_3\}$.

For any vertex v in a valid configuration τ of the six-vertex model, incoming edges can be paired with outgoing edges in exactly two ways, corresponding to two of the three pairings at v. That is to say, τ can be decomposed into $2^{|V|}$ distinct directed Eulerian partitions denoted by $\Phi(\tau)$. Since no two Eulerian orientations share one directed Eulerian partition and every directed Eulerian partition corresponds to a particular Eulerian orientation, the map from six-vertex configurations to directed Eulerian partitions is 1-to-$2^{|V|}$, non-overlapping and surjective. Define w as a function assigning a weight to every pairing at every vertex and let the weight $\widetilde{w}(\varphi)$ of an Eulerian partition φ be the product of weights at each vertex. In particular, when w is defined as

$$\begin{cases} w(\rho_1) = \frac{-a+b+c}{2} \\ w(\rho_2) = \frac{a-b+c}{2} \\ w(\rho_3) = \frac{a+b-c}{2} \end{cases},$$

or equivalently

$$\begin{cases} a = w(\rho_2) + w(\rho_3) \\ b = w(\rho_1) + w(\rho_3) \\ c = w(\rho_1) + w(\rho_2) \end{cases}.$$

for every vertex with the signature $\begin{bmatrix} 0 & 0 & 0 & a \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ a & 0 & 0 & 0 \end{bmatrix}$, then the weight of a six-vertex model configuration τ is equal to $\sum_{\varphi\in\Phi(\tau)} \widetilde{w}(\varphi)$.

2. Markov Chain and Canonical Path

We employ the Holant view to compute the partition function by designing and analyzing a rapid-mixing Markov chain $\mathcal{M}$ to construct an **FPRAS**. Let $G = (V, U, E)$ be the underlying bipartite graph of an instance in Holant$(\neq_2 \mid \mathcal{F}_{\leq^2})$. An assignment σ assigns a value in $\{0, 1\}$ to each edge in E. For every $k \in \mathbb{N}$, we define Ω_k as the collection of assignments which violate $\neq_2$ at exactly k vertices in V. The Markov chain $\mathcal{M}$ is defined on the state space $\Omega = \Omega_0 \cup \Omega_2$.

For every $\sigma \in \Omega$ and any subset $S \subseteq \Omega$, define the *weight function* $\mathcal{W}$ by $\mathcal{W}(\sigma) = \prod_{u\in U} f_u(\sigma|_{E(v)})$ and $\mathcal{Z}(S) = \sum_{\sigma\in S} \mathcal{W}(\sigma)$. Define the Gibbs measure for Ω as $\pi(\sigma) = \frac{\mathcal{W}(\sigma)}{\mathcal{Z}(\Omega)}$. Note that if an assignment $\sigma \in \Omega_2$ assigns 00 to edges incident to $v' \in V$ (satisfying $=_2$ at v'), then it must assign 11 to both edges incident to $v'' \in V$.

Now we describe the transition graph of $\mathcal{M}$. The transition includes three kinds of moves. Suppose that $\sigma \in \Omega_0$. An Ω_0-to-Ω_2 move from σ takes a 4-degree vertex $u \in U$ and two incident edges $e' = (v', u)$, $e'' = (v'', u)$ satisfying $\{\sigma(e'), \sigma(e'')\} = \{0, 1\}$, and changes it to $\sigma_2 \in \Omega_2$ which flips both $\sigma(e')$ and $\sigma(e'')$. An Ω_2-to-Ω_0 move is the opposite. An Ω_2-to-Ω_2 move is, intuitively, to shife one $(=_2)$ from a vertex $v' \in V$ to another $v^* \in V$ where for some $u \in U$, v' and v^* are both incident to u and the "two-0, two-1" rule at u is preserved. Formally, let $\sigma \in \Omega_2$ be the assignment with $v', v'' \in V$ violating $\neq_2$. Let $v^* \in V \setminus \{v', v''\}$ be a vertex in V such that for some $u \in U$, both $e' = (v', u), e^* = (v^*, u) \in E$ and $\{\sigma(e'), \sigma(e^*)\} = \{0, 1\}$. Then an Ω_2-to-Ω_2 move changes σ to σ^* by flipping both $\sigma(e')$ and $\sigma(e^*)$.

If σ_1 can move to σ_2 in the transition graph, we denote by $\sim$ the moves. Note that σ_2 can also move to σ_1. The transition probability $P(\cdot,\cdot)$ is defined as

$$P(\sigma_1,\sigma_2)=\begin{cases}\frac{1}{n^2}\min\left\{1,\frac{\pi(\sigma_2)}{\pi(\sigma_1)}\right\}, & \sigma_2\sim\sigma_1\\ 1-\frac{1}{n^2}\sum_{\sigma'\sim\sigma_1}\min\left\{1,\frac{\pi(\sigma')}{\pi(\sigma_1)}\right\} & \sigma_1=\sigma_2\\ 0 & \text{otherwise}\end{cases}.$$

Since P is designed in a Metropolis way, we know that the stationary distribution of P should be π (if it exists). The following fact comes directly from the definition.

Fact 2.1. *The Markov kernel P is aperiodic, irreducible and reversible with respect to π.*

2.1. **Construction and analysis of canonical path.** To show the rapid mixing of P, we use the method of a flow argument. The key ingredient is to construct a flow with low congestion.

Theorem 2.2 (Lemma 4.2 in [CLL19])**.** *Assume that $\mathcal{Z}(\Omega_0)>0$. There is a flow on Ω with congestion at most $O\left(n^3\left(\frac{\mathcal{Z}(\Omega)}{\mathcal{Z}(\Omega_0)}\right)^2\right)$, using path of length $O(n)$.*

Our goal is to construct the flow $\mathscr{F}:\mathcal{P}\to\mathbb{R}_{\geq 0}$ from Ω_2 to Ω_0 satisfying that

$$\sum_{p\in\mathcal{P}_{\sigma_2,\sigma_0}}\mathscr{F}(p)=\pi(\sigma_2)\pi(\sigma_0),\quad\forall\sigma_2\in\Omega_2,\sigma_0\in\Omega_0$$

where $\mathcal{P}_{\sigma_2,\sigma_0}$ is the collection of all simple directed paths from σ_2 to σ_0 in $\mathcal{M}$ and $\mathcal{P}=\bigcup_{\sigma_2\in\Omega_2,\sigma_0\in\Omega_0}\mathcal{P}_{\sigma_2,\sigma_0}$. With $\mathscr{F}$ in hand, the flow from Ω_0 to Ω_2 can be symmetrically constructed by $\mathscr{F}$. The flow from Ω_2 to Ω_2 or from Ω_0 to Ω_0 can be constructed by randomly picking an intermediate state in Ω_0 or Ω_2.

Now we illustrate the flow $\mathscr{F}$. Let $\Omega'=\Omega_0\cup\Omega_2\cup\Omega_4$. For $\sigma,\sigma'\in\Omega'$, we use $\sigma\oplus\sigma'$ to denote the symmetric difference where we view them as bit strings in $\{0,1\}^E$. We also treat $\sigma\oplus\sigma'$ as an edge subset of E and this induces a subgraph of G. Since at every $u\in U$ of degree 4, the "two-0 two-1" rule is satisfied by σ and σ', this induced subgraph has even degree (0, 2 or 4) at every $u\in U$.

Let $U_4\subseteq U$ be the set of degree-4 vertices in $\sigma\oplus\sigma'$. Then there are exactly $2^{|U_4|}$ Eulerian partitions for $\sigma\oplus\sigma'$. Recall that the Eulerian partition of $\sigma\oplus\sigma'$ is uniquely determined by a family of pairings on U_4. This is a one-to-one correspondence. For any pairing in $\{\rho_1,\rho_2,\rho_3\}$ on a vertex u with signature matrix $M(f)=\begin{bmatrix} & & & a\\ & b & c & \\ & c & b & \\ a & & & \end{bmatrix}$, define the weight function wt for pairings as $\begin{cases}wt(\rho_1)=\frac{-a^2+b^2+c^2}{2}\\ wt(\rho_2)=\frac{a^2-b^2+c^2}{2}\\ wt(\rho_3)=\frac{a^2+b^2-c^2}{2}\end{cases}$ or equivalently $\begin{cases}a^2=wt(\rho_2)+wt(\rho_3)\\ b^2=wt(\rho_1)+wt(\rho_3)\\ c^2=wt(\rho_1)+wt(\rho_2)\end{cases}$. Since $f_u\in\mathcal{F}_{\leq^2}$, all weights take non-negative values. Let $\Phi_{\sigma\oplus\sigma'}$ be the collection of all Eulerian partitions for $\sigma\oplus\sigma'$. For every $\varphi\in\Phi_{\sigma\oplus\sigma'}$, define

$$\mathscr{W}(\sigma,\sigma',\varphi):=\left(\prod_{u\in U\setminus U_4}f_u(\sigma|_{E(u)})f_u(\sigma'|_{E(u)})\right)\left(\prod_{u\in U_4}wt(\varphi(u))\right).$$

Then for all distinct $\sigma, \sigma' \in \Omega'$, we have

$$\begin{aligned}
\sum_{\varphi \in \Phi_{\sigma\oplus\sigma'}} \mathscr{W}(\sigma,\sigma',\varphi) &= \sum_{\varphi \in \Phi_{\sigma\oplus\sigma'}} \left(\prod_{u\in U\setminus U_4} f_u(\sigma|_{E(u)}) f_u(\sigma'|_{E(u)})\right)\left(\prod_{u\in U_4} wt(\varphi(u))\right) \\
&= \left(\prod_{u\in U\setminus U_4} f_u(\sigma|_{E(u)}) f_u(\sigma'|_{E(u)})\right)\left(\sum_{\varphi \in \Phi_{\sigma\oplus\sigma'}} \prod_{u\in U_4} wt(\varphi(u))\right) \\
&= \left(\prod_{u\in U\setminus U_4} f_u(\sigma|_{E(u)}) f_u(\sigma'|_{E(u)})\right)\left(\prod_{u\in U_4} f_u(\sigma|_{E(u)}) f_u(\sigma'|_{E(u)})\right) \\
&= \mathcal{W}(\sigma)\mathcal{W}(\sigma').
\end{aligned}$$

Now we specify the paths in the flow. For a pair of assignments $\sigma_2 \in \Omega_2$ and $\sigma_0 \in \Omega_0$, to transit from σ_2 to σ_0, paths in $\mathcal{P}_{\sigma_2,\sigma_0}$ go through states in Ω that gradually decrease the number of conflicting assignments along walks and circuits in $\sigma_2 \oplus \sigma_0$. We assume an order on E. This induces a total order on circuits in $\sigma_2 \oplus \sigma_0$. By definition, in the induced subgraph $\sigma_2 \oplus \sigma_0$, there are exactly two vertices in V of degree 1 (we call them *endpoints*) and all other vertices are of degree 2 or 4. Note that every path in $\mathcal{P}_{\sigma_2,\sigma_0}$ corresponds to an element in $\Phi_{\sigma_2\oplus\sigma_0}$. Then given any family of pairings $\varphi \in \Phi_{\sigma_2\oplus\sigma_0}$, we have a unique decomposition of the induced subgraph $\sigma_2 \oplus \sigma_0$ as an edge-disjoint union of one walk $[e_1](v_1, e_1', u_1, e_2, v_2, e_2', u_2, \ldots, e_k, v_k)[e_k']$ where e_1, e_k' are not part of the walk, and some edge-disjoint circuits which are ordered lexicographically. Here $v_i \in V$ and $u_i \in U$, and assume that $\sigma_2(e_1) = \sigma_2(e_1') = 0, \sigma_2(e_2) = 1, \sigma_2(e_2') = 0, \ldots, \sigma_2(e_k) = \sigma_2(e_k') = 1$. Thus we know that v_1, v_k satisfy $=_2$. The unique path p_φ firstly "pushes" $=_2$ from v_1 to v_2, then to $v_3, \ldots, v_{k-1}$, and finally "merges" at v_k, arriving at a configuration in Ω_0. Then we reverse all arrows on each circuit in lexicographic order, and within each circuit C it starts at the least edge e and reverses all arrows on C in the direction defined by the starting cyclic orientation of σ_2. Then we make the value of the flow on p_φ be $\frac{\mathscr{W}(\sigma_2,\sigma_0,\varphi)}{\mathcal{Z}(\Omega)^2}$.

Proposition 2.3. *The flow $\mathscr{F} : \mathcal{P} \to \mathbb{R}_{\geq 0}$ defined as above satisfies that*

$$\sum_{p\in\mathcal{P}_{\sigma_2,\sigma_0}} \mathscr{F}(p) = \pi(\sigma_2)\pi(\sigma_0), \quad \forall \sigma_2 \in \Omega_2, \sigma_0 \in \Omega_0.$$

Proof. Note that only the flows on p_φ have non-zero flow. Then we verify that

$$\begin{aligned}
\sum_{p_\varphi\in\mathcal{P}_{\sigma_2,\sigma_0}} \mathscr{F}(p_\varphi) &= \sum_{\varphi\in\Phi_{\sigma_2,\sigma_0}} \frac{\mathscr{W}(\sigma_2,\sigma_0,\varphi)}{\mathcal{Z}(\Omega)^2} \\
&= \frac{\mathcal{W}(\sigma_2)\mathcal{W}(\sigma_0)}{\mathcal{Z}(\Omega)^2} \\
&= \pi(\sigma_2)\pi(\sigma_0).
\end{aligned}$$

□

Lemma 2.4. *The flow $\mathscr{F}$ has congestion at most $O(n^3)\frac{\mathcal{Z}(\Omega_2)}{\mathcal{Z}(\Omega_0)}$.*

Proof. For any transition $(\sigma', \sigma'') \in \mathcal{M}$ where $\sigma' \neq \sigma''$, we bound $P(\sigma', \sigma'')$ by

$$P(\sigma',\sigma'') = \frac{1}{n^2}\min\left\{1, \frac{\pi(\sigma'')}{\pi(\sigma')}\right\} = \Omega\left(n^{-2}\right)$$

since the quantity $\frac{\pi(\sigma'')}{\pi(\sigma')}$ is a constant. Let

$$H_{\sigma'} := \left\{\sigma_2 \oplus \sigma_0 \,\middle|\, \sigma_2 \in \Omega_2, \sigma_0 \in \Omega_0, \exists \varphi \in \Phi_{\sigma_2\oplus\sigma_0}, \sigma' \in p_\varphi\right\}.$$

We bound the congestion ρ of $\mathcal{F}$ as

$$\begin{aligned}
\rho &= \max_{(\sigma',\sigma'')\in\mathcal{M}} \frac{1}{\pi(\sigma')P(\sigma',\sigma'')} \sum_{\sigma_2\in\Omega_2,\sigma_0\in\Omega_0} \sum_{p_\varphi\in\mathcal{P}_{\sigma_2,\sigma_0},p_\varphi\ni(\sigma',\sigma'')} \frac{\mathscr{W}(\sigma_2,\sigma_0,\varphi)}{\mathcal{Z}(\Omega)^2}\\
&\le \max_{\sigma'\in\Omega} \frac{O(n^2)}{\mathcal{W}(\sigma')\mathcal{Z}(\Omega)} \sum_{\sigma\in\Omega_2,\sigma_0\in\Omega_0} \sum_{\varphi\in\Phi_{\sigma_2,\sigma_0},p_\varphi\ni\sigma'} \mathscr{W}(\sigma_2,\sigma_0,\varphi)\\
&\le \max_{\sigma'\in\Omega} \frac{O(n^2)}{\mathcal{W}(\sigma')\mathcal{Z}(\Omega)} \sum_{\sigma_2\in\Omega_2} \sum_{\eta\in H_{\sigma'}} \sum_{\varphi\in\Phi_\eta} \mathscr{W}(\sigma_2,\sigma_2\oplus\eta,\varphi).
\end{aligned}$$

Now we fix $\sigma'\in\Omega$. For any $\sigma_2\in\Omega_2$ and $\eta\in H_{\sigma'}$ consisting of exactly one connected component with two endpoints of degree 1 and all other vertices having even degrees, observe that $\sigma'\oplus\eta\in\Omega'$. Note that if $\sigma'\in\Omega_0$ then $\sigma'\oplus\eta\in\Omega_2$; if $\sigma'\in\Omega_2$, then depending on whether σ'

(1) is σ_2, or
(2) appears in the process of reversing arrows on the walk with two endpoints, or
(3) appears after reversing arrows on the walk with endpoints,

the assignment $\sigma'\oplus\eta$ is in Ω_0, Ω_2 or Ω_4 respectively. Note that

$$\mathscr{W}(\sigma_2,\sigma_2\oplus\eta,\varphi) = \left(\prod_{u\in U\setminus U_4} f_u(\sigma_2|_{E(u)})f_u((\sigma_2\oplus\eta)|_{E(u)})\right)\left(\prod_{u\in U_4} wt(\varphi(u))\right).$$

For every degree-0 vertex $u\in U$, f_u takes the same value in $\sigma_2,\sigma_2\oplus\eta,\sigma'$ and $\sigma'\oplus\eta$. Otherwise, for every 2-degree vertex $u\in U$, $f_u(\sigma_2|_{E(u)}), f_u((\sigma_2\oplus\eta)|_{E(u)})$ take two different values in $\{a,b,c\}$. Similarly $f_u(\sigma'|_{E(u)}), f_u((\sigma'\oplus\eta)|_{E(u)})$ also take two these different values in $\{a,b,c\}$. Then we know that $\mathscr{W}(\sigma_2,\sigma_2\oplus\eta,\varphi)=\mathscr{W}(\sigma',\sigma'\oplus\eta,\varphi)$. Then we can show that

$$\begin{aligned}
\rho &\le \max_{\sigma'\in\Omega} \frac{O(n^2)}{\mathcal{W}(\sigma')\mathcal{Z}(\Omega)} \sum_{\sigma_2\in\Omega_2} \sum_{\eta\in H_{\sigma'}} \sum_{\varphi\in\Phi_\eta} \mathscr{W}(\sigma_2,\sigma_2\oplus\eta,\varphi)\\
&\le \max_{\sigma'\in\Omega} \frac{O(n^2)|E|}{\mathcal{W}(\sigma')\mathcal{Z}(\Omega)} \sum_{\eta\in H_{\sigma'}} \sum_{\varphi\in\Phi_\eta} \mathscr{W}(\sigma',\sigma'\oplus\eta,\varphi)\\
&\le \max_{\sigma'\in\Omega} \frac{O(n^3)}{\mathcal{W}(\sigma')\mathcal{Z}(\Omega)} \sum_{\eta\in H_{\sigma'}} \mathcal{W}(\sigma'\oplus\eta)\\
&\le O(n^3)\frac{\mathcal{Z}(\Omega')}{\mathcal{Z}(\Omega)}
\end{aligned}$$

with a standard argument $\mathcal{Z}(\Omega_4)/\mathcal{Z}(\Omega_2)\le\mathcal{Z}(\Omega_2)/\mathcal{Z}(\Omega_0)$. Therefore, the congestion is bounded by $O(n^3)\frac{\mathcal{Z}(\Omega_2)}{\mathcal{Z}(\Omega_0)}$. □

2.2. **Windability in six-vertex models.**

In [McQ13, HLZ16], a standard way to establish an **FPRAS** for Holant problems is to show the *windability* of signatures.

Definition 2.5 (Windability)**.** For any finite set J and any configuration $x\in\{0,1\}^J$, define $\mathcal{M}_x$ as the set of partitions of $\{i\in J\mid x_i=1\}$ into pairs and at most one singleton. We say a signature $f:\{0,1\}^J\to\mathbb{R}_{\ge0}$ is *windable* if there exists values $B(x,y,M)\ge0$ for any distinct $x,y\in\{0,1\}^J$ and $M\in\mathcal{M}_{x\oplus y}$ satisfying that

- $f(x)f(y)=\sum_{M\in\mathcal{M}_{x\oplus y}} B(x,y,M)$ for any distinct $x,y\in\{0,1\}^J$;
- $B(x,y,M)=B(x\oplus S,y\oplus S,M)$ for all distinct $x,y\in\{0,1\}^J$ and $S\in M\in\mathcal{M}_{x\oplus y}$.

Lemma 2.6 (Windability of $\mathcal{F}_{\leq 2}$)**.** *For any nonnegative real numbers a, b, c, the function f with signature matrix* $M(f) = \begin{bmatrix} & & & a \\ & b & c & \\ & c & b & \\ a & & & \end{bmatrix}$ *is windable if and only if $a^2 \leq b^2 + c^2$, $b^2 \leq a^2 + c^2$ and $c^2 \leq a^2 + b^2$.*

3. Hardness

By now, the intractability of the six-vertex model is consistent with what has been established in physics.

Theorem 3.1 (Theorem 5.1 in [CLL19])**.** *If $f \in \mathcal{F}_{>}$, then* $\mathrm{Holant}(\neq_2 | f)$ *does not have an* **FPRAS** *unless* **RP** = **NP**.

References

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