COUPLING FOR BLOCK DYNAMICS AND k-HEIGHTS

1. k-heights Models and Two Random Walks

In statistical physics, the *k-heights* is a great important model for studying the energy of gas models. We use a combinatorial language to express it. Given a graph $G = (V, E)$ and a positive integer k, a k-height is an assignment $\sigma : V \to \{0, 1, \ldots, k\}$ such that for every $e = (u, v) \in E$, $|\sigma(u) - \sigma(v)| \le 1$. Let $\Omega = \Omega_{G,k}$ be the set of all k-heights on G.

We study how to sample from the uniform distribution on Ω. We use $\mathcal{U}(\cdot)$ to denote the uniform distribution on the state space. The method we apply is a Markov chain named the *up-down random walk*. Let $\mathcal{M}$ denote the dynamics. A transition of $\mathcal{M}$ depends on a vertex $v \in V$ and a signature $\Delta \in \{-1, +1\}$ which direct the developing of the value on v. We describe this random walk as Algorithm 1.

Algorithm 1: a transition step of the up-down random walk $\mathcal{M}$;

input : an initial assignment $X_t \in \Omega$ in the Markov chain;
output : a final assignment $X_{t+1} \in \Omega$ meaning a step of transition from X_t;

sample $v \in V$, $\Delta \in \{-1, +1\}$ and $p \in [0, 1]$ uniformly at random;
if $p \le \frac{1}{2}$ **then**
  $X_{t+1} \leftarrow X_t$;
else
  define the assignment $\sigma : V \to \{0, 1, \ldots, k\}$ as

$$\sigma(u) := \begin{cases} X_t(u) + \Delta & u = v \\ X_t(u) & u \neq v \end{cases};$$

  if σ *is a valid k-height* **then**
    $X_{t+1} \leftarrow \sigma$;
  else
    $X_{t+1} \leftarrow X_t$;
return X_{t+1}.

The aperiodicity and ergodicity of $\mathcal{M}$ come directly from the definition. It can be easily shown that the detailed balanced equation holds and hence $\mathcal{M}$ is reversible with respect to the uniform distribution on Ω.

Unfortunately, it is not easy to analyze directly the mixing rate of $\mathcal{M}$. Instead, we analyze the block dynamics $\mathcal{M}_{\mathcal{B}}$ and use *Markov chain comparison* to show the rapid mixing of $\mathcal{M}$. We firstly fix a family of blocks $\mathcal{B} = \{B_1, \ldots, B_\ell\}$ covering V, *i.e.*, $\cup_{i=1}^{\ell} B_i = V$. For a block $B \in \mathcal{B}$, we define the boundary ∂B as the set $\partial B := \{v \in V \setminus B \mid \exists u \in B, (u, v) \in E\}$. We denote by Ω_B the set of k-heights of the subgraph of G induced by B, *i.e.*,

$$\Omega_B := \{\sigma : B \to \{0, 1, \ldots, k\} \mid \sigma \text{ } k\text{-height w.r.t. } G[B]\}.$$

For $X \in \Omega$ and $\sigma : B \to \{0, 1, \ldots, k\}$, we define the assignment $[X|\sigma] : V \to \{0, 1, \ldots, k\}$ as the assignment which maps $v \in B$ to $\sigma(v)$ and $v \in V \setminus B$ to $X(v)$.

Now we are ready to define *admissible fillings* of B in X. We denote the set of all these admissible fillings by $\Omega_{B|X}$ as

$$\Omega_{B|X} := \{\sigma \in \Omega_B \mid [X|\sigma] \in \Omega\}.$$

Note that $\Omega_{B|X}$ satisfies a kind of Markov property such that $\Omega_{B|X}$ and $\Omega_{B|X'}$ are same when X corresponds to X' at ∂B. We then safely extend the definition of $\Omega_{B|X}$ to k-heights X only defined on ∂B. We call such a $X \in \Omega_{\partial B}$ a *boundary constraint.* A boundary constraint $X \in \Omega_{\partial B}$ is *extensible* if $\Omega_{B|X} \neq \emptyset$.

The block dynamics $\mathcal{M}_{\mathcal{B}}$ could be seen as an extension of the up-down random walk $\mathcal{M}$. At each transition step, we pick a block uniformly at random and update it to obtain the next assignment. We formally state $\mathcal{M}_{\mathcal{B}}$ in Algorithm 2.

Algorithm 2: a transition step of the block dynamics $\mathcal{M}_{\mathcal{B}}$;

input : an initial assignment $X_t \in \Omega$ in the Markov chain;
output : a final assignment $X_{t+1} \in \Omega$ meaning a step of transition from X_t;

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sample B ∈ 𝓑, σ ∈ Ω_{B|X_t} and p ∈ [0,1] uniformly at random;
if p ≤ 1/2 then
    X_{t+1} ← X_t;
else
    X_{t+1} ← [X_t|σ];
return X_{t+1}.
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For convenience, we use $\mathcal{E}(\cdot)$ to denote the transitions in a Markov scheme.

1.1. **Path coupling.** A main technique to show the rapid mixing of the Markov dynamics is the *coupling of Markov chains.* The main ingredient of this classical method is to construct a proper coupling for two Markov chains $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ and then show the contraction of such a coupling.

Definition 1 (Contraction of coupling)**.** Fix a state space Ω and a metric d on Ω. Let $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ be two chains induced by the Markov kernel $\mathcal{C}$ on Ω and $\gamma = (X_t, Y_t)_{t\geq 0}$ be a coupling of them. We say γ is *α-contractive with* d for a factor $\alpha < 1$ if for every $t \geq 0$, it holds that

$$\mathbf{E}\left[d(X_{t+1}, Y_{t+1}) \mid X_t, Y_t\right] \leq \alpha d(X_t, Y_t).$$

Usually, we choose the Hamming distance $d_H(\cdot,\cdot)$ as the metric d. The mixing rate of a Markov chain $\mathcal{C}$ can be upper bounded by the factor α.

Theorem 2. *Let $\gamma = (X_t, Y_t)_{t\geq 0}$ be a coupling of a Markov kernel $\mathcal{C}$ on the state space Ω. Suppose that γ is α-contractive with a metric d on Ω. Define the diameter of Ω with d as*

$$d_{\max} := \max_{x,y\in\Omega} d(x,y).$$

The the mixing rate $\tau(\mathcal{C},\varepsilon)$ of $\mathcal{C}$ can be upper bounded by

$$\tau(\mathcal{C},\varepsilon) \leq \frac{\log(d_{\max}/\varepsilon)}{1-\alpha}.$$

Unfortunately, it is often hard to show the contraction of a coupling directly. For instance, calculating the decrement/increment after one transition for any pair of (X_t, Y_t) is not often easy. To overcome this difficulty, we employ the *path coupling theorem* which allows us to focus only on pairs in a much smaller subset.

Theorem 3. *Suppose that $\mathcal{C}$ is a Markov chain on Ω and $d : \Omega \times \Omega \to \mathbb{N}$ is a metric on Ω. Furthermore, suppose that there exists a subset $S \subseteq \Omega \times \Omega$ such that for every $(x, y) \in \Omega$, there exists a path $x = x_0, x_1, \ldots, x_k = y$ such that*

$$(x_i, x_{i+1}) \in S, \forall i = 0, 1, \ldots, k-1 \quad \text{and} \quad \sum_{i=0}^{k-1} d(x_i, x_{i+1}) = d(x, y).$$

If for every $(X_t, Y_t) \in S$, there exists a coupling $(X_t, Y_t) \to (X_{t+1}, Y_{t+1})$ and a factor $\alpha < 1$ such that

$$\mathbf{E}\left[d(X_{t+1}, Y_{t+1}) \mid X_t, Y_t\right] \leq \alpha d(X_t, Y_t),$$

then this coupling can be extended to an α-contractive coupling on the whole Ω.

1.2. **Relationship between $\mathcal{M}$ and $\mathcal{M}_{\mathcal{B}}$.** We state here how to derive the mixing rate of $\mathcal{M}$ with the mixing rate of $\mathcal{M}_{\mathcal{B}}$ in hand. This comparison technique is the so-called "canonical path" introduced by Jerrum and Sinclair [JS96], and we use a version stated in Randall and Tetali [RT98].

Theorem 4 (Theorem 3 in [RT98])**.** *Let $\mathcal{C}$ and $\widetilde{\mathcal{C}}$ be two reversible Markov chains on the same state space Ω and having the same stationary distribution π. Let $E(\mathcal{C})$ be the set of transitions of $\mathcal{C}$ and $E(\widetilde{\mathcal{C}})$ be the set of transitions in $\widetilde{\mathcal{C}}$.*

Suppose that for each transition $(x, y) \in E(\widetilde{\mathcal{C}})$, there is a path $\gamma_{x,y} : x = x_0, \ldots, x_k = y$ of transitions (x_i, x_{i+1}) in $E(\mathcal{C})$. For a transition $(u, v) \in \mathcal{C}$, let

$$\Gamma(u, v) := \left\{(x, y) \in E(\widetilde{\mathcal{C}}) \;\middle|\; (u, v) \in \gamma_{x,y}\right\}.$$

Define the quantity

$$A := \max_{(u,v)\in E(\mathcal{C})} \frac{1}{\pi(u)\mathcal{C}(u, v)} \sum_{(x,y)\in\Gamma(x,y)} |\gamma_{x,y}|\pi(x)\widetilde{\mathcal{C}}(x, y)$$

where $|\gamma_{x,y}|$ is the length of the path $\gamma_{x,y}$ and the quantities $\mathcal{C}(u, v) = \mathbf{Pr}_{\mathcal{C}}\left[X_{t+1} = v \mid X_t = u\right]$, $\widetilde{\mathcal{C}}(x, y) = \mathbf{Pr}_{\widetilde{\mathcal{C}}}\left[X_{t+1} = y \mid X_t = x\right]$ are transition probabilities in Markov chains $\mathcal{C}, \widetilde{\mathcal{C}}$ respectively. Then for every $\varepsilon \in (0, 1)$, the mixing rate $\tau(\mathcal{C}, \varepsilon)$ of $\mathcal{C}$ can be upper bounded by the mixing rate $\tau(\widetilde{\mathcal{C}}, \varepsilon)$ as

$$\tau(\mathcal{C}, \varepsilon) \leq \frac{4\log(1/(\varepsilon \cdot \pi_{\min}))}{\log(1/2\varepsilon)} \cdot A \cdot \tau(\widetilde{\mathcal{C}}, \varepsilon)$$

where $\pi_{\min} = \min_{x\in\Omega} \pi(x)$.

2. Monotone Coupling and Rapid Mixing of $\mathcal{M}$ and $\mathcal{M}_{\mathcal{B}}$

Now we construct a *monotone coupling* for (X_t, Y_t) to apply the path coupling theorem. However, the existence of such a monotone coupling is not trivial to see. We establish some properties of k-heights and the block dynamics at first and construct a monotone coupling according to these properties. In the end, we analyze the contraction of this monotone coupling and prove the mixing rate of $\mathcal{M}_{\mathcal{B}}$ and leading to the rapid mixing of $\mathcal{M}$.

2.1. **Properties of k-heights and block divergence.** The most important property of the block dynamics is the *stochastic dominance*. We offer a partial order on Ω defined as $X \leq Y$ if $X(v) \leq Y(v)$ for every $v \in V$. We say $U \subseteq \Omega$ is an *upset* if $x \in U$ then $y \in U$ for every $x \leq y \in \Omega$. For two probability distributions μ_1, μ_2 on Ω, we say μ_1 *stochastically dominated* by μ_2 if $\mu_1(U) \leq \mu_2(U)$ for every upset $U \subseteq \Omega$.

The following theorem can be viewed as a discrete version of a theorem by Strassen.

Theorem 5. *Let μ_1 and μ_2 be two probability distributions over Ω such that μ_1 is stochastically dominated by μ_2. Then there exists a joint distribution λ of μ_1 and μ_2 on $\Omega \times \Omega$ satisfying that if $\lambda(x,y) > 0$, then $x \le y$.*

To describe the mixing rate of $\mathcal{M}_{\mathcal{B}}$, the following concept named *block divergence* plays a critical role. We call a pair $(X,Y) \in \Omega \times \Omega$ a *cover pair* if $X \le Y$ and $d_H(X,Y) = 1$. That is to say, X and Y only differ at exactly one vertex $v \in V$ and $Y(v) = X(v) + 1$. Let $B \in \mathcal{B}$ be some block. If $v \in \partial B$, the sets of admissible fillings $\Omega_{B|X}$ and $\Omega_{B|Y}$ might be different. The uniform distributions $\mathcal{U}(\Omega_{B|X})$ and $\mathcal{U}(\Omega_{B|Y})$ too. We view $\mathcal{U}(\Omega_{B|X})$ and $\mathcal{U}(\Omega_{B|Y})$ as two distributions on Ω_B. Then by Theorem 5 there exists a joint distribution $\lambda_{B,X,Y}$ on $\Omega_B \times \Omega_B$, which is exactly a joint distribution on $\Omega_{B|X} \times \Omega_{B|Y}$.

To show the rapid mixing of $\mathcal{M}_{\mathcal{B}}$, when the next state (X',Y') is drawn from $\lambda_{B,X,Y}$, the expectation of $d(X',Y')$ is of great importance. Define the *block divergence* $E_{B,v}$ for every $B \in \mathcal{B}$ and $v \in \partial B$ as

$$E_{B,v} := \max_{(X,Y)\in\Omega\times\Omega \text{ a cover pair}, Y(v)=X(v)+1} \mathbf{E}_{(X',Y')\sim\lambda_{B,X,Y}}\left[d_H(X',Y')\right].$$

An immediate question is how to compute $\mathbf{E}_{(X',Y')\sim\lambda_{B,X,Y}}\left[d_H(X',Y')\right]$. The following lemma gives an answer. For an admissible filling $\sigma \in \Omega_B$, let $w(\sigma) := \sum_{v\in V}\sigma(v)$ be its weight.

Lemma 6. *Let $(X,Y) \in \Omega \times \Omega$ be a cover pair and $B \in \mathcal{B}$ be some block. Then it holds that*

$$\mathbf{E}_{(X',Y')\sim\lambda_{B,X,Y}}\left[d_H(X',Y')\right] = \mathbf{E}_{\sigma\sim\mathcal{U}(\Omega_{B|Y})}\left[w(\sigma)\right] - \mathbf{E}_{\sigma\sim\mathcal{U}(\Omega_{B|X})}\left[w(\sigma)\right].$$

The proof of Lemma 6 comes directly from the definition after noting that $X' \le Y'$ by Theorem 5.

2.2. **Stochastic dominance in block chains.** Now we state the most important property of $\mathcal{U}(\Omega_{B|X})$ and $\mathcal{U}(\Omega_{B|Y})$ when (X,Y) is a cover pair. We will make use of the Ahlswede-Daykin *four functions theorem.*

Lemma 7 (Four functions theorem)**.** *Let D be a distributive lattice and $f_1, f_2, f_3, f_4 : D \to \mathbb{R}_{\ge 0}$ such that for all $a, b \in D$,*

$$f_1(a)f_2(b) \le f_3(a \vee b)f_4(a \wedge b).$$

Then for all $A, B \subseteq D$,

$$f_1(A)f_2(B) \le f_3(A \vee B)f_4(A \wedge B)$$

where $f_i(A) = \sum_{a\in A} f_i(a)$, $A \vee B = \{a \vee b \mid a \in A, b \in B\}$ and $A \wedge B = \{a \wedge b \mid a \in A, b \in B\}$.

Lemma 8. *Let $X, Y \in \Omega, X \le Y$ be two k-heights of $G = (V,E)$ and $B \subseteq V$ be a block. Let D be the smallest distributive sublattice of Ω_B containing $\Omega_{B|X} \cup \Omega_{B|Y}$. Then $\Omega_{B|X}$ forms a downset and $\Omega_{B|Y}$ forms an upset.*

Proof. By symmetry, we prove that $\Omega_{B|X}$ forms a downset in D. That is to say, for $g, h \in D$, $g \le h$, if $h \in \Omega_{B|X}$ then $g \in \Omega_{B|X}$.

Suppose that $g \notin \Omega_{B|X}$. Since $g \in \Omega_B$, we must have $|g(v) - X(v')| \ge 1$ for two adjacent vertices $v \in B$ and $v' \in \partial B$. Since we know $h \in \Omega_{B|X}$, it holds that $g(v) \le h(v) \le X(v') + 1$ so that

$$g(v) < X(v') - 1.$$

For every $f \in \Omega_{B|X}$, we know that $f(v) \ge X(v') - 1$ by definition. Also for every $f \in \Omega_{B|Y}$ we have $f(v) \ge Y(v') - 1$ and thus $f(v) \ge X(v') - 1$ by $X \le Y$. Therefore,

$$\min_{f\in D} f(v) = \min\left\{f(v) \;\middle|\; f \in \Omega_{B|X} \cup \Omega_{B|Y}\right\} \ge X(v') - 1 > g(v).$$

This leads to a contradiction to $g \in D$. $\square$

Then we are ready to introduce stochastic dominance in block chains.

Proposition 9. *Let $X, Y \in \Omega$, $X \leq Y$ be two k-heights of $G = (V, E)$ and $B \subseteq V$ be a block. Then $\mathcal{U}(\Omega_{B|X})$ is stochastic dominated by $\mathcal{U}(\Omega_{B|Y})$ on Ω_B.*

Proof. Let D be the smallest distributive lattice of $\Omega_{B|X} \cup \Omega_{B|Y}$. We consider $\mathcal{U}(\Omega_{B|X})$ and $\mathcal{U}(\Omega_{B|Y})$ on D. Then we want to show that for every upset $U \subseteq D$,

$$0 \leq \mathcal{U}(\Omega_{B|Y})(U) - \mathcal{U}(\Omega_{B|X})(U) = \frac{\left|U \cap \Omega_{B|Y}\right|}{\left|\Omega_{B|Y}\right|} - \frac{\left|U \cap \Omega_{B|X}\right|}{\left|\Omega_{B|X}\right|}.$$

We define the four functions as

$$f_1(h) := \mathbb{1}\left[h \in U \cap \Omega_{B|X}\right], f_2(h) := \mathbb{1}\left[h \in U \cap \Omega_{B|Y}\right],$$
$$f_3(h) := \mathbb{1}\left[h \in U \cap \Omega_{B|Y}\right], f_4(h) := \mathbb{1}\left[h \in U \cap \Omega_{B|X}\right].$$

We aim to verify that for every $g, h \in D$,

$$f_1(h) f_2(g) \leq f_3(h \vee g) f_4(h \wedge g).$$

When $f_1(h) f_2(g) = 0$, the inequality holds trivially. Then we assume that $f_1(h) = f_2(g) = 0$, *i.e.*, $h \in U \cap \Omega_{B|X}$ and $g \in \Omega_{B|Y}$. By properties from Lemma 8, we conclude that $f_3(h \vee g) = f_4(h \wedge g) = 1$. Then by Lemma 7, it holds that

$$0 \leq f_3(D) f_4(D) - f_1(D) f_2(D) = \left|U \cap \Omega_{B|Y}\right| \cdot \left|\Omega_{B|X}\right| - \left|U \cap \Omega_{B|X}\right| \cdot \left|\Omega_{B|Y}\right|.$$

Then we yield the inequality we need. □

2.3. **Monotone coupling for block dynamics.** Now we are ready to construct a monotone coupling for $\mathcal{M}_{\mathcal{B}}$. To apply Theorem 3, we define

$$\Omega' := \{(X, Y) \in \Omega \times \Omega \mid (X, Y) \text{ is a cover pair}\}.$$

Then we only construct the monotone coupling on S and extend it to the whole space.

Algorithm 3: monotone coupling $(X_t, Y_t)_{t \geq 0}$ of $\mathcal{M}_{\mathcal{B}}$;

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input : a pair of $(X_t, Y_t) \in \Omega \times \Omega$;
output : a pair of $(X_{t+1}, Y_{t+1}) \in \Omega \times \Omega$ meaning a step of monotone coupling;
sample $p \in [0, 1]$ uniformly at random;
if $p \leq \frac{1}{2}$ **then**
    $(X_{t+1}, Y_{t+1}) \leftarrow (X_t, Y_t)$;
else
    if $d_H(X_t, Y_t) \leq 1$ **then**
        sample $B \in \mathcal{B}$ uniformly at random;
        if $X_t(v) = Y_t(v)$ *for all* $v \in \partial B$ **then**
            sample $\sigma \sim \mathcal{U}(\Omega_{B|X_t})$;
            $(X_{t+1}, Y_{t+1}) \leftarrow ([X_t|\sigma], [Y_t|\sigma])$;
        else
            obtain $\lambda = \lambda_{B, X_t, Y_t}$ by Theorem 5;
            sample $(\sigma_X, \sigma_Y) \sim \lambda$;
            $(X_{t+1}, Y_{t+1}) \leftarrow ([X_t|\sigma_X], [Y_t|\sigma_Y])$;
    else
        define (X_{t+1}, Y_{t+1}) using path coupling theorem;
return (X_{t+1}, Y_{t+1}).

It is not hard to verify that the coupling we construct is a proper coupling of $\mathcal{M}_{\mathcal{B}}$ on Ω. Also by Theorem 5, the coupling is trivially monotone.

Lemma 10. *Define the quantity α as for every $v \in V$,*

$$1 - \frac{1}{2|\mathcal{B}|}\left(|B \in \mathcal{B} \mid v \in B| - \sum_{B\in\mathcal{B}, v\in\partial B}(E_{B,v}-1)\right) \le \alpha.$$

Then for every $(X,Y) \in \Omega'$ and the transition (X',Y') after the monotone coupling, it holds that

$$\mathbf{E}\left[d_H(X',Y') \;\middle|\; (X,Y)\right] \le \alpha d_H(X,Y).$$

Proof. Let $Y(v) = X(v)+1$ for some $v \in V$ and $X(w) = Y(w)$ for every $w \in V \setminus \{v\}$. We study the case $p \ge \frac{1}{2}$ since the distance does not change when $p \le 1/2$.

When $v \in B$, then $X(w) = Y(w)$ for every $w \in \partial B$. By definition, we know that $X' = Y'$ and thus $d_H(X',Y') = 0$. When $v \in \partial B$, by the definition of $E_{B,v}$, it holds that

$$\mathbf{E}\left[d_H(X',Y') \;\middle|\; v \in \partial B\right] \le E_{B,v}.$$

When $v \notin (B \cup \partial B)$, then we know that $d_H(X',Y') = 1$. Putting all things together, we obtain

$$\begin{aligned}\mathbf{E}\left[d_H(X',Y')\right] &\le \frac{1}{2} + \frac{1}{2}\left(\frac{\sum_{B\in\mathcal{B}, v\in\partial B} E_{B,v}}{|B|} + \left(1 - \frac{|B \in \mathcal{B} \mid v \in B \vee v \in \partial B|}{|B|}\right)\right)\\ &= 1 - \frac{1}{2|\mathcal{B}|}\left(|B \in \mathcal{B} \mid v \in B| - \sum_{B\in\mathcal{B}, v\in\partial B}(E_{B,v}-1)\right) \le \alpha.\end{aligned}$$

□

Then we conclude the main result of k-heights on $G = (V,E)$.

Theorem 11. *Let $G = (V,E)$ be a finite graph and $\mathcal{B}$ be a finite family of blocks such that $\cup_{B\in\mathcal{B}} B = V$. If there exists a factor $\alpha < 1$ such that for all $v \in V$,*

$$1 - \frac{1}{2|\mathcal{B}|}\left(|B \in \mathcal{B} \mid v \in B| - \sum_{B\in\mathcal{B}, v\in\partial B}(E_{B,v}-1)\right) \le \alpha,$$

then for the mixing time $\tau(\mathcal{M},\varepsilon)$ of the up-down random walk on k-heights of G, we have

$$\tau(\mathcal{M},\varepsilon) \le c_{\mathcal{B},k} \cdot \frac{\left(|V|\log(1/\varepsilon) + |V|^2\log(k+1)\right)\cdot\log(k|V|/\varepsilon)}{\log(1/2\varepsilon)}$$

where $m := \max_{v\in V}|B \in \mathcal{B} \mid v \in B|$ and $b := \max_{B\in\mathcal{B}}|B|$ and

$$c_{\mathcal{B},k} := \frac{8 \cdot bmk(k+1)^b}{(1-\alpha)|\mathcal{B}|}.$$

Proof. Since $1 - \frac{1}{2|\mathcal{B}|}\left(|B \in \mathcal{B} \mid v \in B| - \sum_{B\in\mathcal{B}, v\in\partial B}(E_{B,v}-1)\right) \le \alpha$, by Lemma 10 and Theorem 3, it holds that the monotone coupling is α-contractive on Ω. Therefore, by Theorem 2, the mixing rate $\tau(\mathcal{M}_{\mathcal{B}},\varepsilon)$ of the block dynamics $\mathcal{M}_{\mathcal{B}}$ can be upper bounded by

$$\tau(\mathcal{M}_{\mathcal{B}},\varepsilon) \le \frac{\log(k|V|/\varepsilon)}{1-\alpha}$$

together with the observation $d_{\max} = k \cdot |V|$.

To bound the mixing rate of $\mathcal{M}$, we use the Markov chain comparison Theorem 4. Let $b := \max_{B\in\mathcal{B}}|B|$ and let $X \to Y$ be a transition in $\mathcal{M}_{\mathcal{B}}$. Then there exists a block $B \in \mathcal{B}$ such that the two k-heights X and Y only differ at B. Note that there exists a *shortest* path $\gamma_{X,Y}$ of length $|\gamma_{X,Y}| = d_H(X,Y) \leq k|B| \leq k \cdot b$ using transitions in $\mathcal{M}$. We could only choose $\gamma_{X,Y}$ of length $d_H(X,Y)$ since only values on vertices in B will change.

For every $B \in \mathcal{B}$, we use the notation $\mathcal{M}_{\mathcal{B}}(\cdot,\cdot \mid B)$ to denote the transition probabilities conditional on $p \geq \frac{1}{2}$ and the picked transition block $B \in \mathcal{B}$. For $X \neq Y$, by the law of total probability, it holds that

$$\mathcal{M}_{\mathcal{B}}(X,Y) = \frac{1}{2|\mathcal{B}|}\sum_{B\in\mathcal{B}} \mathcal{M}_{\mathcal{B}}(X,Y \mid B).$$

Now fix some (W,Z) in transitions of $\mathcal{M}$ which is not a loop. Let v be the vertex on which W and Z differ. Then the probability is

$$\mathcal{M}(W,Z) = \frac{1}{4|V|}.$$

We consider the set

$$\Gamma(W,Z) := \{(X,Y) \in \mathcal{E}(\mathcal{M}_{\mathcal{B}}) \mid (W,Z) \in \gamma_{X,Y}\}.$$

Let $(X,Y) \in \Gamma(W,Z)$ and B be a block with $\mathcal{M}_{\mathcal{B}}(X,Y \mid B) > 0$. Observe that X and Y only differ at B. Moreover, since $(W,Z) \in \gamma_{X,Y}$, it holds that $X(v) \neq Y(v)$ and $v \in B$. For any arbitrary block $B \in \mathcal{B}$, observe that

$$\sum_{(X,Y)\in\Gamma(W,Z)} \mathcal{M}_{\mathcal{B}}(X,Y \mid B) \leq \sum_{X',Y'\in\Omega_B} \frac{1}{|\Omega_B|} = |\Omega_B| \leq (k+1)^b.$$

Then we know that

$$\begin{aligned}\sum_{(X,Y)\in\Gamma(W,Z)} \mathcal{M}_{\mathcal{B}}(X,Y) &\leq \frac{1}{2|\mathcal{B}|}\sum_{B\in\mathcal{B},v\in B}\ \sum_{(X,Y)\in\Gamma(W,Z)} \mathcal{M}_{\mathcal{B}}(X,Y \mid B)\\ &\leq \frac{|\{B\in\mathcal{B} \mid v \in B\}|}{2|\mathcal{B}|}(k+1)^b.\end{aligned}$$

Now we consider the quantity $A(W,Z)$ defined as

$$A(W,Z) := \frac{1}{\pi(W)\mathcal{M}(W,Z)}\sum_{(X,Y)\in\Gamma(W,Z)} |\Gamma_{X,Y}|\pi(X)\mathcal{M}_{\mathcal{B}}(X,Y).$$

Plugging all bounds into it, we obtain that

$$A(W,Z) \leq 2|V| \cdot bk(k+1)^b \frac{|\{B\in\mathcal{B} \mid v\in B\}|}{|\mathcal{B}|}.$$

Then we obtain the desired upper bound of the mixing rate $\tau(\mathcal{M},\varepsilon)$. □

The following corollary is easier to use.

Corollary 12. *Let $G = (V,E)$ be a finite graph and $\mathcal{B}$ be a family of blocks such that $\cup_{B\in\mathcal{B}} B = V$. Furthermore, assume that for every vertex $v \in V$, it occurs in at most m blocks and in at most s boundaries of blocks. Let $E_{\max} := \max_{B\in\mathcal{B},v\in\partial B} E_{B,v}$. If there exists $\alpha < 1$ such that*

$$1 - \frac{1}{2|\mathcal{B}|}(m - s\cdot(E_{\max}-1)) \leq \alpha,$$

then the mixing rate $\tau(\mathcal{M},\varepsilon)$ of the up-down random walk has an upper bound stated in Theorem 11.

References

[JS96] Mark Jerrum and Alistair Sinclair. The Markov chain Monte Carlo method: an approach to approximate counting and integration. In *Approximation Algorithms for NP-Hard Problems*, page 482–520, USA, 1996. PWS Publishing Co. 3

[RT98] Dana Randall and Prasad Tetali. Analyzing Glauber dynamics by comparison of Markov chains. In Cláudio L. Lucchesi and Arnaldo V. Moura, editors, *LATIN'98: Theoretical Informatics*, pages 292–304, Berlin, Heidelberg, 1998. Springer Berlin Heidelberg. 3