COMPUTATIONAL TREE FOR MULTI-SPIN SYSTEMS

1. Multi-spin Systems

In the field of statistical physics, multi-spin systems are a family of models describing the energy of interactions between spins. We study on graphic multi-spin models, namely the Ising model (2-spin) or the Potts model (q-spin for $q \geq 3$).

Definition 1.1 (Multi-spin systems)**.** A *multi-spin* system is a tuple $\mathcal{S} = (G, q, \{\boldsymbol{\lambda}_v\}_{v\in V}, \{A_e\}_{e\in E})$ where $G = (V, E)$ is a graph, $q \geq 2$ is a positive integer denoting the number of spins, $\boldsymbol{\lambda}_v \in \mathbb{R}^q$ is a vector representing the energy of each spin for each $v \in V$ and $A_e \in \mathbb{R}^{q\times q}_{\geq 0}$ is a symmetric matrix representing the energy of interactions between spins for each edge $e \in E$.

The energy or weight of a configuration $\sigma \in [q]^V$ is defined by

$$w_{\mathcal{S}}(\sigma) := \prod_{v\in V} \boldsymbol{\lambda}_v(\sigma(v)) \prod_{\{u,v\}\in E} A_{\{u,v\}}(\sigma(u), \sigma(v))$$

The partition function of the system which denotes the energy of the whole system is defined by

$$Z_{\mathcal{S}} := \sum_{\sigma\in[q]^V} w_{\mathcal{S}}(\sigma) = \sum_{\sigma\in[q]^V} \prod_{v\in V} \boldsymbol{\lambda}_v(\sigma(v)) \prod_{\{u,v\}\in E} A_{\{u,v\}}(\sigma(u), \sigma(v)). \tag{1}$$

Another important term is the Gibbs distribution of $\mathcal{S}$. Let $\mu = \mu_{\mathcal{S}}$ be the Gibbs distribution defined by

$$\mu_{\mathcal{S}}(\sigma) := \frac{w(\sigma)}{Z_{\mathcal{S}}}, \quad \forall \sigma \in [q]^V.$$

1.1. Notations.

At first, for a graph $G = (V, E)$, we give an order to V. Then there exists a natural order on E extended by the order on V.

An instance of a q-spin system with a pinning is a tuple $(\mathcal{S}, \Lambda, \sigma_\Lambda)$ where $\Lambda \subseteq V$ is the frozen vertex set and $\sigma_\Lambda \in [q]^\Lambda$. The frozen vertex set Λ together with σ_Λ is called a pinning. We note here that σ_Λ encodes the information of the frozen vertices Λ and simply use σ_Λ to denote the pinning. The weights, partition function and Gibbs distribution of Φ are defined similarly to the multi-spin system denoted by $w^{\sigma_\Lambda}_{\mathcal{S}}$, $Z^{\sigma_\Lambda}_{\mathcal{S}}$ and $\mu^{\sigma_\Lambda}_{\mathcal{S}}$ defined on $[q]^{V\setminus\Lambda}$. We say σ_Λ is a *feasible* pinning if $Z^{\sigma_\Lambda}_{\mathcal{S}} > 0$. We only consider feasible pinnings to ensure the Gibbs distribution is well-defined. Let Ω^{σ_Λ} be the support of $\mu^{\sigma_\Lambda}_{\mathcal{S}}$. For a subset $B \subseteq V \setminus \Lambda$, we use $\mu^{\sigma_\Lambda}_{\mathcal{S},B}$ to denote the marginal distribution of $\mu^{\sigma_\Lambda}_{\mathcal{S}}$ projected to B conditional on σ_Λ. When $\mathcal{S}$ is clear, we drop the script $\mathcal{S}$.

2. Self-avoiding Walk Tree and Correlation-decay Tree

To estimate the partition function of a multi-spin system $\mathcal{S}$, Weitz [Wei06] proposed a structure named the *self-avoiding walk tree* for the case $q = 2$. Nair and Tetali [NT07] generalized the self-avoiding walk tree to the *correlation-decay tree* for general multi-spin systems. A crucial property motivating the two structures is the recursion for the marginal ratio. For an instance $(\mathcal{S}, \Lambda, \sigma_\Lambda)$ and a vertex $v \in V$, the marginal ratio of v is defined by

$$R^{\sigma_\Lambda}_{\mathcal{S},v}(\pi_v, \rho_v) := \frac{\mu^{\sigma_\Lambda}_{\mathcal{S},v}(\pi_v)}{\mu^{\sigma_\Lambda}_{\mathcal{S},v}(\rho_v)}, \quad \forall \pi_v, \rho_v \in \Omega^{\sigma_\Lambda}_v. \tag{2}$$

For an instance $(\mathcal{S} = (G = (V, E), q, \{\boldsymbol{\lambda}_v\}_{v\in V}, \{A_e\}_{e\in E}), \Lambda, \sigma_\Lambda)$ and an unfrozen vertex $v \in V \setminus \Lambda$, let $u_1, \ldots, u_m$ be the neighbors of v in G in order. We define a new graph G_v by making m copies $\{v_1, \ldots, v_m\}$ of v and each v_i has exactly a single edge to u_i. Fix two different feasible spins $\pi_v, \rho_v \in \Omega^{\sigma_\Lambda}_v$ on v. For each $i = 1, \ldots, m$, define the partial assignment $(\rho_v \oplus \pi_v)^{(i)}$ by assigning π_v to $v_1, \ldots, v_{i-1}$ and ρ_v to $v_{i+1}, \ldots, v_m$. Then, for each $i = 1, \ldots, m$, define a new instance $\left(\mathcal{S}^{(i)}, \Lambda^{(i)}, \sigma_{\Lambda^{(i)}}\right)$ as following:

- $\mathcal{S}^{(i)} = \left(G^{(i)} = (V^{(i)}, E^{(i)}), q, \left\{\lambda_u^{(i)}\right\}_{u \in V^{(i)}}, \left\{A_e^{(i)}\right\}_{e \in E^{(i)}}\right)$ where $G^{(i)} = G_v \setminus \{v_i\}$, $\lambda_u^{(i)} = \lambda_u$ for $u \neq v_j$ and $\lambda_{v_j}^{(i)} = \lambda_{v_j}^{1/m}$ for each $j \neq i$, and $A_e^{(i)} = A_e$ for $e \neq \{v_i, u_i\}$ and $A_{\{v_i,u_i\}}^{(i)} = A_{\{v,u_i\}}$;
- $\Lambda^{(i)} = \Lambda \cup \{v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_m\}$;
- $\sigma_{\Lambda^{(i)}} = \sigma_\Lambda \cup (\rho_v \oplus \pi_v)^{(i)}$.

The following lemma illustrates the recursive form of the marginal ratio.

Lemma 2.1 ([NT07]). *For a multi-spin instance $(\mathcal{S}, \Lambda, \sigma_\Lambda)$, a vertex $v \in V \setminus \Lambda$ and $\pi_v, \rho_v \in \Omega_v^{\sigma_\Lambda}$, the following recursion holds for the marginal ratio $R_{\mathcal{S},v}^{\sigma_\Lambda}(\pi_v, \rho_v)$:*

$$R_{\mathcal{S},v}^{\sigma_\Lambda}(\pi_v, \rho_v) = \frac{\lambda_v(\pi_v)}{\lambda_v(\rho_v)} \prod_{i=1}^{m} \frac{\sum_{c=1}^{q} A_{\{v,u_i\}}(\pi_v, c) R_{\mathcal{S}^{(i)},u_i}^{\sigma_{\Lambda^{(i)}}}(c, \rho_v)}{\sum_{c=1}^{q} A_{\{v,u_i\}}(\rho_v, c) R_{\mathcal{S}^{(i)},u_i}^{\sigma_{\Lambda^{(i)}}}(c, \rho_v)}$$

Proof. Consider the following multi-spin system $\mathcal{S}' = \left(G' = (V', E'), \left\{\lambda'_u\right\}_{u \in V'}, \{A_e\}_{e \in E'}\right)$ defined by:

- $G' = G_v$;
- $\lambda'_{v_j} = \lambda_{v_j}^{1/m}$ for every $j = 1, \ldots, m$ and $\lambda'_u = \lambda_u$ for $u \neq v_j$;
- $A'_e = A_e$ for $e \neq \{v_i, u_i\}$ and $A'_{\{v_i,u_i\}} = A_{\{v,u_i\}}$.

Observe that

$$\frac{\mathbb{P}_{\mathcal{S}}\left(\sigma(v) = \pi_v \mid \sigma_\Lambda\right)}{\mathbb{P}_{\mathcal{S}}\left(\sigma(v) = \rho_v \mid \sigma_\Lambda\right)} = \frac{\mathbb{P}_{\mathcal{S}'}\left(\sigma(v_1) = \pi_v, \ldots, \sigma(v_m) = \pi_v \mid \sigma_\Lambda\right)}{\mathbb{P}_{\mathcal{S}'}\left(\sigma(v_1) = \rho_v, \ldots, \sigma(v_m) = \rho_v \mid \sigma_\Lambda\right)}$$

Then we consider the following marginal ratios:

$$R_i = \frac{\mathbb{P}_{\mathcal{S}'}\left(\sigma(v_1) = \pi_v, \ldots, \sigma(v_{i-1}) = \pi_v, \sigma(v_i) = \pi_v, \sigma(v_{i+1}) = \rho_v, \ldots, \sigma(v_m) = \rho_v \mid \sigma_\Lambda\right)}{\mathbb{P}_{\mathcal{S}'}\left(\sigma(v_1) = \pi_v, \ldots, \sigma(v_{i-1}) = \pi_v, \sigma(v_i) = \rho_v, \sigma(v_{i+1}) = \rho_v, \ldots, \sigma(v_m) = \rho_v \mid \sigma_\Lambda\right)}.$$

Trivially it holds that $R_{\mathcal{S},v}^{\sigma_\Lambda}(\pi_v, \rho_v) = \prod_{i=1}^{m} R_i$.

For each $i = 1, \ldots, m$, by definition it holds that R_i is equal to the marginal ratio $R_{\mathcal{S}',v_i}^{\sigma_{\Lambda^{(i)}}}(\pi_v, \rho_v)$ of $\sigma(v_i) = \pi_v$ and $\sigma(v_i) = \rho_v$ conditional on σ_Λ and $(\pi_v \oplus \rho_v)^{(i)}$. Then we enumerate the value of $\sigma(u_i)$ and remove the edge $\{v_i, u_i\}$, obtaining that

$$\begin{aligned} R_{\mathcal{S}',v_i}^{\sigma_{\Lambda^{(i)}}}(\pi_v, \rho_v) &= \frac{\lambda'_{v_i}(\pi_v)}{\lambda'_{v_i}(\rho_v)} \frac{\sum_{c=1}^{q} A'_{\{v,u_i\}}(\pi_v, c) R_{\mathcal{S}^{(i)},u_i}^{\sigma_{\Lambda^{(i)}}}(c, \rho_v)}{\sum_{c=1}^{q} A'_{\{v,u_i\}}(\rho_v, c) R_{\mathcal{S}^{(i)},u_i}^{\sigma_{\Lambda^{(i)}}}(c, \rho_v)} \\ &= \frac{\lambda_{v_i}(\pi_v)^{1/m}}{\lambda_{v_i}(\rho_v)^{1/m}} \frac{\sum_{c=1}^{q} A_{\{v,u_i\}}(\pi_v, c) R_{\mathcal{S}^{(i)},u_i}^{\sigma_{\Lambda^{(i)}}}(c, \rho_v)}{\sum_{c=1}^{q} A_{\{v,u_i\}}(\rho_v, c) R_{\mathcal{S}^{(i)},u_i}^{\sigma_{\Lambda^{(i)}}}(c, \rho_v)}. \end{aligned}$$

Hence we obtain the desired result. □

Remark 2.2. We remark here that when T is a rooted tree and we aim to estimate the marginal ratio at the root v, the recursion is of the following form: suppose that v has m subtrees $T_1, \ldots, T_m$, then

$$R_{\mathcal{T},v}^{\sigma_\Lambda}(\pi_v, \rho_v) = \frac{\lambda_v(\pi_v)}{\lambda_v(\rho_v)} \prod_{i=1}^{m} \frac{\sum_{c=1}^{q} A_{\{v,u_i\}}(\pi_v, c) R_{\mathcal{T}_i,u_i}^{\sigma_{\Lambda_i}}(c, \rho_v)}{\sum_{c=1}^{q} A_{\{v,u_i\}}(\rho_v, c) R_{\mathcal{T}_i,u_i}^{\sigma_{\Lambda_i}}(c, \rho_v)}$$

where Λ_i includes the frozen vertices in the subtree T_i fixed to ρ_v when we fix the spin at v to be π_v.

2.1. **Weitz's computational tree and correlation-decay tree.** With Lemma 2.1 in hand, Nair and Tetali [NT07] proposed a computational tree named the *correlation-decay tree* to estimate the partition function of a multi-spin system. However, instead of introducing the correlation-decay tree directly, we first introduce Weitz's computational tree [Wei06] which can be viewed as a special case of the correlation-decay tree when $q = 2$.

For a graph $G = (V, E)$ and a vertex $v \in V$, remember that we have an order on E. We define a self-avoiding walk tree $T_{\text{SAW}}(G, v)$ rooted at v as following.

Definition 2.3 (Self-avoiding walk tree [Wei06])**.** For a graph $G = (V, E)$ and a vertex $v \in V$, the self-avoiding walk tree $T = T_{\mathrm{SAW}}(G, v)$ is a tree rooted at $r = r(G, v)$ where each internal node is associated with a self-avoiding walk starting from v and each leaf is associated with a self-avoiding walk with no expansion or a self-avoiding walk with a edge closing a cycle on G satisfying that node w is the father of w' if and only the walk from w' is extended by w with a single-step walk.

Moreover, we label all leaves representing cycles of T as occupied or unoccupied according to the following law: for a leaf, we say it is occupied if and only if the edge closing the cycle is larger than that starting the cycle.

For a node w in the self-avoiding walk tree $T = T_{\mathrm{SAW}}(G, v)$, it is natural to identify w with a vertex v^w on the original graph G. Fix a 2-spin system $\mathcal{S} = (G = (V, E), 2, \{\lambda_v\}_{v\in V}, \{A_e\}_{e\in E})$ and a vertex $v \in V$. We extend the 2-spin system $\mathcal{S}$ to the self-avoiding walk tree from v on G. Let $\mathcal{T}_{\mathrm{SAW}}(\mathcal{S}, v) := \left(T = (V(T), E(T)), 2, \{\lambda^T_w\}_{w\in V(T)}, \{A^T_e\}_{e\in E(T)}\right)$ be the 2-spin system generated from $\mathcal{S}$ by:

- $T = T_{\mathrm{SAW}}(G, v)$ be the self-avoiding walk tree from v on G;
- $\lambda^T_w = \lambda_{v^w}$ for each $w \in V(T)$;
- $A^T_{\{w,z\}} = A_{\{v^w, z^w\}}$ for each $\{w, z\} \in E(T)$.

For a pinning σ_Λ, we extend it to a pinning on T by fixing all nodes identified with $u \in \sigma_\Lambda$ with the corresponding value in σ_Λ.

The following lemma shows that T_{SAW} preserves the marginal ratio of the Gibbs distribution of a 2-spin system.

Lemma 2.4 ([Wei06])**.** *Fix a 2-spin system $\mathcal{S} = (G = (V, E), 2, \{\lambda_v\}_{v\in V}, \{A_e\}_{e\in E})$. For an instance $(\mathcal{S}, \Lambda, \sigma_\Lambda)$ where $\Lambda \subseteq V$ is a frozen vertex subset and σ_Λ is a feasible pinning and a vertex $v \in V \setminus \Lambda$, let $\mathcal{T} = \mathcal{T}_{\mathrm{SAW}}(\mathcal{S}, v)$ be the 2-spin system extended by $\mathcal{S}$ and v. Then it holds that*

$$\mu^{\sigma_\Lambda}_{\mathcal{T}}(\sigma(v) = 1) = \mu^{\sigma_\Lambda}_{\mathcal{S}}(\sigma(v) = 1).$$

Lemma 2.4 can be shown easily from the recursion on $T_{\mathrm{SAW}}(G, v)$ and the recursion on the original spin system. For simplicity we omit it here.

In [Wei06], the author raised a question whether the method of the self-avoiding walk tree could be generalized to multi-spin systems. Nair and Tetali [NT07] illustrates that the very difference between two-spin systems and multi-spin systems lies in the recursion that in the binary spin model, one of the spin systems was always the reference spin and the other was the subject of this recursion. To overcome this barrier, they generalize the construction of $T_{\mathrm{SAW}}(G, v)$ and use a similar structure named *correlation-decay tree* by adding some *coupling lines* on the self-avoiding walk tree.

Definition 2.5 (Coupling line [NT07])**.** A *coupling line* on a rooted tree is a virtual line connecting a vertex u to some vertex v in the subtree rooted at u. This line works as following: when one descends into the subtree rooted at u with assumed spin c, we freeze v with the spin c.

Definition 2.6 (Correlation-decay tree [NT07])**.** Given a positive integer $q \geq 2$, fix a multi-spin system $\mathcal{S} = (G = (V, E), q, \{\lambda_v\}_{v\in V}, \{A_e\}_{e\in E})$ and a vertex $v \in V$. Let $T = T_{\mathrm{SAW}}(G, v)$ be the self-avoiding walk tree from v on G. For a feasible reference spin c, we define the *correlation-decay tree* $T_{\mathrm{CD}}(G, v, c)$ from T_{SAW} by drawing coupling line from occupied leaves to their ancestors with the same identified vertex on T and associating all unoccupied leaves with spin c.

We define the instance $\mathcal{T}_{\mathrm{CD}}(\mathcal{S}, v, c)$ in a similar way to $\mathcal{T}_{\mathrm{SAW}}(\mathcal{S}, v)$. Similar to Lemma 2.4, $T_{\mathrm{CD}}(G, v, c)$ preserves the marginal ratio.

Lemma 2.7 ([NT07])**.** *Fix a spin system $\mathcal{S} = (G = (V, E), q, \{\lambda_v\}_{v\in V}, \{A_e\}_{e\in E})$. For an instance $(\mathcal{S}, \Lambda, \sigma_\Lambda)$ where $\Lambda \subseteq V$ is a frozen vertex subset and σ_Λ is a feasible pinning, a vertex $v \in V \setminus \Lambda$ and a feasible reference spin $\rho_v \in \Omega^{\sigma_\Lambda}_v$, let $\mathcal{T} = \mathcal{T}_{\mathrm{CD}}(\mathcal{S}, v, \rho_v)$ be the q-spin system extended by $\mathcal{S}$, v and ρ_v. Then it holds that*

$$R^{\sigma_\Lambda}_{\mathcal{T},v}(\pi_v, \rho_v) = R^{\sigma_\Lambda}_{\mathcal{S},v}(\pi_v, \rho_v), \quad \forall \pi_v \in \Omega^{\sigma_\Lambda}_v.$$

3. Spatial Mixing by Correlation-decay Tree

To estimate the partition function $Z_{\mathcal{S}}$, the following spatial mixing property is crucial.

Definition 3.1 (Strong/weak spatial mixing)**.** The Gibbs distribution μ of a multi-spin system $\mathcal{S} = (G = (V, E), q, \{\lambda_v\}_{v\in V}, \{A_e\}_{e\in E})$ is said to have the *strong spatial mixing* property if there exists a constant $C > 0$ such that for every $\Lambda \subseteq V$, for each $v \in V \setminus \Lambda$ and any pair of feasible pinnings $\sigma_\Lambda, \tau_\Lambda$ on Λ, it holds that

$$\left\|\mu_v^{\sigma_\Lambda} - \mu_v^{\tau_\Lambda}\right\|_{\mathrm{TV}} \leq \exp(-C \cdot \mathrm{dist}_G(v, \Gamma)) \tag{3}$$

where $\Gamma \subseteq \Lambda$ stands for the subset of vertices that σ_Λ and τ_Λ differ.

Similarly, the Gibbs distribution μ of a multi-spin system $\mathcal{S} = (G = (V, E), q, \{\lambda_v\}_{v\in V}, \{A_e\}_{e\in E})$ is said to have the *weak spatial mixing* property if there exists a constant $C > 0$ such that for every $\Lambda \subseteq V$, for each $v \in V \setminus \Lambda$ and any pair of feasible pinnings $\sigma_\Lambda, \tau_\Lambda$ on Λ, it holds that

$$\left\|\mu_v^{\sigma_\Lambda} - \mu_v^{\tau_\Lambda}\right\|_{\mathrm{TV}} \leq \exp(-C \cdot \mathrm{dist}_G(v, \Lambda)). \tag{4}$$

To relate the strong spatial mixing on the original graph G and that on the correlation-decay tree $T_{\mathrm{CD}}(G, v, c)$, we formally introduce the following concepts for a collection of *valid coupling lines* and the *very strong spatial mixing* on trees.

Fix a rooted tree T with its root r. For a collection L of virtual edges on T, we say L is a collection of *valid coupling lines* if L satisfies that: each line in L joins a node to some node in the subtree under it; the lower endpoints of the coupling lines are unique; no pair of lines form a nested pair or an interleaved pair, *i.e.*, the endpoints do not lie on a single path.

Definition 3.2 (Very strong spatial mixing on trees [NT07])**.** Fix a rooted tree T with its root r. Let $\delta : \mathbb{N} \to \mathbb{R}_{>0}$ be a decreasing function tending to 0 as $n \to \infty$. We say the distribution over the spin system at the root v of T exhibits the *very strong spatial mixing with rate* δ if and only if for every feasible spin π_v at v, every set of legal coupling lines L on T, every subset $\Lambda \subseteq V(T)$ of nodes and any two feasible pinnings $\sigma_\Lambda, \tau_\Lambda$ on Λ, it holds that

$$|\mathbb{P}_T(\sigma(v) = \pi_v \mid \sigma_\Lambda, L) - \mathbb{P}_T(\sigma(v) = \pi_v \mid \tau_\Lambda, L)| \leq \delta(\mathrm{dist}_T(v, \Gamma)) \tag{5}$$

where $\Gamma \subseteq \Lambda$ is the set of nodes where σ_Λ and τ_Λ differ.

Some observations from statistical physics have shown that, when we investigate the uniqueness of the Gibbs distribution on infinite graphs of maximum degree d, the infinite regular tree $\mathbb{T}_d$ of maximum degree d becomes the hardest case. In detail, the infinite regular tree $\mathbb{T}_d$ is a rooted tree with its root v satisfying that each node except v has exactly $d-1$ children and the root has d ones. We consider the case that $\lambda_u = \lambda$ for all $u \in V$ and $A_e = A$ for all $\{u, v\} \in E$. For convenience, we assume a special spin 0 with $\lambda(0) = \gamma > 0$ and $A(0, c) = \kappa > 0$ for each $c = 0, \dots, q+1$.

From the view of spatial mixing, the following lemma shows that the very strong spatial mixing on the infinite regular tree $\mathbb{T}_d$ with an extra spin 0 implies the strong spatial mixing on graphs of maximum degree d with same interactions.

Theorem 3.3 (Theorem 3.10 in [NT07])**.** *Fix a positive integer $d \geq 2$. If the infinite regular tree $\mathbb{T}_d$ exhibits the very strong spatial mixing with rate δ for the spin system with $\lambda_u = \lambda$ for all $u \in V$ and $A_e = A$ for all $e \in E$ and a special spin 0 with $\lambda(0) = \alpha > 0$ and $A(0, c) = \beta > 0$, then the Gibbs distribution on graphs of maximum degree d with the same interactions exhibits the strong spatial mixing with rate δ.*

Proof. Let T_Λ is the tree generated from $T_{\mathrm{CD}}(G, v, q)$ with a frozen vertex subset Λ. That is to say, when we meet a frozen vertex, we stop the growth of the tree and prune it. Note that the tree T_Λ does not depend on the pinning on Λ. Then with a same manner in the proof of Lemma 2.7, it holds that:

$$|\mathbb{P}_G(\sigma(v) = \pi_v \mid \sigma_\Lambda) - \mathbb{P}_G(\sigma(v) = \pi_v \mid \tau_\Lambda)| = \left|\mathbb{P}_{T_\Lambda}(\sigma(v) = \pi_v \mid \sigma_\Lambda) - \mathbb{P}_{T_\Lambda}(\sigma(v) = \pi_v \mid \tau_\Lambda)\right|$$

for any two feasible configuration $\sigma_\Lambda, \tau_\Lambda$ on Λ.

Assume that Λ includes nodes pinned to q. Note that we can view T_Λ as a subtree of $\mathbb{T}_d$. Let $\partial(T_\Lambda)$ be the non-fixed boundary of T_Λ, *i.e.*, nodes in T_Λ that are not pinned by Λ, are not the lower endpoints of a dotted line, and have degree strictly less than $b+1$. Let Λ_1 be the set of nodes in $\mathbb{T}_d \setminus T_\Lambda$ that is attached to one of the nodes in $\partial(T_\Lambda)$. Then we append Λ_1 to $\mathbb{T}_\Lambda$ to obtain a subtree $\mathbb{T}_d^\Lambda$ of $\mathbb{T}$. We fix the spins on Λ_1 to 0. Then by calculation, it holds that

$$\mathbb{P}_{T_\Lambda}\left(\sigma(v) = \pi_v \mid \sigma_\Lambda\right) = \mathbb{P}_{\mathbb{T}_d^\Lambda}\left(\sigma(v) = \pi_v \mid \sigma_\Lambda \wedge \Lambda_1 \leftarrow \mathbf{0} \wedge T_\Lambda \not\leftarrow 0\right)$$

where the event $T_\Lambda \not\leftarrow 0$ means that nodes in T_Λ do not receive the spin 0. Then by the very strong spatial mixing on $\mathbb{T}_d$ with rate δ, it holds that

$$\begin{aligned}
&\left|\mathbb{P}_G\left(\sigma(v) = \pi_v \mid \sigma_\Lambda\right) - \mathbb{P}_G\left(\sigma(v) = \pi_v \mid \tau_\Lambda\right)\right| \\
&= \left|\mathbb{P}_{T_\Lambda}\left(\sigma(v) = \pi_v \mid \sigma_\Lambda\right) - \mathbb{P}_{T_\Lambda}\left(\sigma(v) = \pi_v \mid \tau_\Lambda\right)\right| \\
&= \left|\mathbb{P}_{\mathbb{T}_d^\Lambda}\left(\sigma(v) = \pi_v \mid \sigma_\Lambda \wedge \Lambda_1 \leftarrow \mathbf{0} \wedge T_\Lambda \not\leftarrow 0\right) - \mathbb{P}_{\mathbb{T}_d^\Lambda}\left(\sigma(v) = \pi_v \mid \tau_\Lambda \wedge \Lambda_1 \leftarrow \mathbf{0} \wedge T_\Lambda \not\leftarrow 0\right)\right| \\
&\leq \delta(\mathrm{dist}_T(v, \Gamma)).
\end{aligned}$$

□

References

[NT07] Chandra Nair and Prasad Tetali. The correlation decay (CD) tree and strong spatial mixing in multi-spin systems, 2007. 1, 2, 3, 4

[Wei06] Dror Weitz. Counting independent sets up to the tree threshold. In *Proceedings of the Thirty-Eighth Annual ACM Symposium on Theory of Computing*, STOC '06, pages 140–149, New York, NY, USA, 2006. Association for Computing Machinery. 1, 2, 3